Mem. Gra. Sci. Eng. Shimane Univ.
Series B: Mathematics
46 (2013), pp. 1–13

SIZE-STRUCTURED POPULATION MODELS HAVING DIFFERENT NONLOCAL TERMS IN VITAL RATES

KOSUKE KAMIZAKI AND NOBUYUKI KATO

(Received: January 31, 2013)

Abstract. We study a system of size-structured population models having nonlinear vital rates such as growth, mortality and fertility rates, each of which has a nonlocal term different from each other. Our aim is to show how can be applied Banach's fixed point theorem to obtain the existence of a unique solution.

1. Introduction

We are concerned with size structured population models with growth rate depending on the individual's size and the weighted total population. Suppose that there are N species and let $p^i(s,t)$ represent the density of population with respect to size $s \in (0, s^i_\dagger)$ at time $t \in [0,T]$ for the i-th species, where $s^i_\dagger \in (0,\infty]$ is the maximum size. It is natural to think that each population interacts in some sense each other. We employ three weighted total populations $P^i_w(t)$, $P^i_m(t)$ and $P^i_b(t)$ with weight functions $w^i(s)$, $m^i(s)$ and $b^i(s)$, respectively, and we assume that vital rates such as growth rate, mortality rate and fertility rate depend on the differently weighted total populations.

Our model describing the dynamics of N-populations is formulated as the following system of initial boundary value problems with different nonlocal terms in vital rates:

$$
\text{(P)} \quad \begin{cases} \partial_t p^i + \partial_s (g^i(s, P_w(t))p^i) = -\mu^i(s, P_m(t))p^i(s,t), & s \in [0, s^i_\dagger),\ t \in [0,T], \\ g^i(0, P_w(t))p^i(0,t) = \displaystyle\int_0^{s^i_\dagger} \beta^i(s, P_b(t))p^i(s,t)\,ds, & t \in [0,T], \\ p^i(s,0) = p^i_0(s), \quad s \in [0, s^i_\dagger), \end{cases}
$$

2010 *Mathematics Subject Classification.* 35F61, 35Q92, 92D25.

Key words and phrases. size-structured population, nonlinear vital rate, weighted total population, unique existence.

where

$$P_w(t) = (P_w^1(t), \cdots, P_w^N(t)),\ P_w^i(t) = \int_0^{s_\dagger^i} w^i(s)p^i(s,t)\,ds, \tag{1}$$

$$P_m(t) = (P_m^1(t), \cdots, P_m^N(t)),\ P_m^i(t) = \int_0^{s_\dagger^i} m^i(s)p^i(s,t)\,ds, \tag{2}$$

$$P_b(t) = (P_b^1(t), \cdots, P_b^N(t)),\ P_b^i(t) = \int_0^{s_\dagger^i} b^i(s)p^i(s,t)\,ds, \tag{3}$$

respectively. Calsina and Saldaña [2] studied a single species model and the usual total population or biomass are considered as the weighted total populations. Their technique is based on reducing to a system of Volterra integral equations as developed for age-structured Gurtin-MacCamy models [3]. Ackleh, Banks, and Deng [1] considered a system of subpopulation model where the birth process is replaced by

$$g^i(0,P(t))p^i(0,t) = C^i(t) + \sum_{j=1}^N \int_0^{s_\dagger} \beta^{ij}(s,P(t))p^j(s,t)\,ds, \tag{4}$$

where $P(t)$ is the usual total population, i.e., $P(t) = P_w(t)$ with $w \equiv 1$ in (1) and $C^i(t)$ represents the inflow of zero-size individuals (i.e. newborns) from outside. They showed existence of a unique weak solution by finite difference approximation technique. It is possible to replace the birth process in (P) to (4) in our analysis but we do not treat such a birth process for simplicity. Kato [4] studied a similar system as (P) but the growth rates are assumed to be common for each species and the methods are based on a system with time-dependent linear growth rate and Schauder's fixed point theorem. Our methods are based on the argument of [4], but in this paper, we show that Banach's fixed point theorem works and obtain the existence of a unique solution.

The paper is organized as follows. In Section 2, we state our assumptions, preliminary facts and the main result. We give some lemmas in Section 3 and prove the main theorem in Section 4.

2. Preliminaries and results

In this section, we first state our assumptions and preliminary facts including definition of solutions. Then we state our main results on the existence of a unique solution of (P). Let $s_\dagger = \max\{s_\dagger^1, \cdots, s_\dagger^N\}$ and $L^1 := L^1(0, s_\dagger; \mathbb{R}^N)$ be the Banach space of Lebesgue integrable functions from $(0, s_\dagger)$ to $\mathbb{R}^N$ with norm $\|\phi\|_{L^1} := \int_0^{s_\dagger} |\phi(s)|_N\,ds = \sum_{i=1}^N \int_0^{s_\dagger^i} |\phi^i(s)|\,ds$ for $\phi \in L^1$, where $|\cdot|_N$ denotes the norm of $\mathbb{R}^N$. Then define $L_0^1 := \{\phi = (\phi^1, \cdots, \phi^N) \in L^1 \mid \phi^i(s) = 0 \quad a.e.\ s \in (s_\dagger^i, s_\dagger)\}$. For $T > 0$, we set $L_T := C([0,T]; L_0^1)$, the Banach space of L^1-valued continuous functions on $[0,T]$ with supremum norm $\|p\|_{L_T} := \sup_{0\le t\le T} \|p(t)\|_{L^1}$ for $p \in L_T$. Note that each element of L_T can be viewed as an element of $L^1((0,s_\dagger)\times(0,T); \mathbb{R}^N)$ by relation $[p^i(t)](s) = p^i(s,t)$ for a.e. $(t,s) \in (0,T)\times(0,s_\dagger)$. See [6, Lemma 2.1]. Furthermore, let $\mathbb{R}_+^N$ be the usual positive cone in $\mathbb{R}^N$, $L_{0,+}^1 := \{\phi \in L_0^1 \mid \phi(s) \in$

$\mathbb{R}^N_+$ for *a.e.* $s \in (0, s_\dagger)\}$, and $L_{T,+} := C([0,T]; L^1_{0,+})$. Finally, let $W^{1,\infty}(0, s_\dagger)$ be the usual Sobolev space. For $i = 1, \cdots, N$, we assume the following basic assumptions:

(H1) $\mu^i : [0, s^i_\dagger) \times \mathbb{R}^N \to \mathbb{R}_+$ is bounded by $\bar{\mu} > 0$ and there is an increasing function $c_\mu : [0, \infty) \to [0, \infty)$ such that

$$|\mu^i(s_1, P_1) - \mu^i(s_2, P_2)| \le c_\mu(r)\,(|s_1 - s_2| + |P_1 - P_2|_N)$$

for $s_1,\ s_2 \in [0, s^i_\dagger)$ and $|P_1|_N,\ |P_2|_N \le r$.

(H2) $\beta^i : [0, s^i_\dagger) \times \mathbb{R}^N \to \mathbb{R}_+$ is bounded by $\bar{\beta} > 0$ and there is an increasing function $c_\beta : [0, \infty) \to [0, \infty)$ such that

$$|\beta^i(s, P_1) - \beta^i(s, P_2)| \le c_\beta(r)\,(|s_1 - s_2| + |P_1 - P_2|_N)$$

for $s_1,\ s_2 \in [0, s^i_\dagger)$ and $|P_1|_N,\ |P_2|_N \le r$.

(H3) $g^i : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}_+$ is a bounded continuous function. $g^i(s, P) > 0$ for $(s, P) \in [0, s^i_\dagger)$ and in case of $s^i_\dagger < \infty$, $g^i(s, P) = 0$ for $(s, P) \in [s^i_\dagger, \infty) \times \mathbb{R}^N$. For each $P \in \mathbb{R}^N$, $g^i(s, P)$ is differentiable with respect to $s \in [0, s^i_\dagger)$ and the partial derivative $\partial_s g^i(s, P)$ is continuous on $[0, s^1_\dagger) \times \mathbb{R}^N$. There exists an increasing function $c_g : [0, \infty) \to [0, \infty)$ such that

$$|g^i(s_1, P_1) - g^i(s_2, P_2)| \le c_g(r)\,(|s_1 - s_2| + |P_1 - P_2|_N)$$

for $s_1,\ s_2 \in [0, s^i_\dagger)$ and $|P_1|_N,\ |P_2|_N \le r$.

(H4) $w^i,\ m^i,\ b^i \in W^{1,\infty}(0, s_\dagger)$ and $0 \le w^i(s) \le \bar{w},\ 0 \le m^i(s) \le \bar{m},\ 0 \le b^i(s) \le \bar{b}$ for some constants $\bar{w},\ \bar{m},\ \bar{b} > 0$.

We may extend the function $g^i(s, P)$ on $(-\infty, \infty) \times \mathbb{R}^N$ keeping the Lipschitz property in (H3) by putting $g^i(s, P) := g^i(0, P)$ for $s \in (-\infty, 0)$. In what follows, $g^i(s, P)$ is supposed to be extended on $(-\infty, \infty) \times \mathbb{R}^N$ as above.

Let $P \in C([0,T]; \mathbb{R}^N)$ be given arbitrarily. Before considering problem (P), we consider the following nonautonomous problem:

$$(\tilde{\mathrm{P}})\quad \begin{cases} \partial_t \tilde{p}^i + \partial_s(g^i(s, P(t))\tilde{p}^i) = -\mu^i(s, \tilde{P}_m(t))\tilde{p}^i(s,t), & s \in [0, s^i_\dagger),\ t \in [0,T], \\ g^i(0, P(t))\tilde{p}^i(0,t) = \displaystyle\int_0^{s^i_\dagger} \beta^i(s, \tilde{P}_b(t))\tilde{p}^i(s,t)\,ds, & t \in [0,T], \\ \tilde{p}^i(s, 0) = p^i_0(s), & s \in [0, s^i_\dagger), \end{cases}$$

where $\tilde{P}_m(t)$ and $\tilde{P}_b(t)$ are defined similarly to $P_m(t)$ and $P_b(t)$ as in (2) and (3).

For given $P \in C([0,T]; \mathbb{R}^N)$, we define the characteristic curve $\varphi^i_P(t; t_0, s_0)$ through $(s_0, t_0) \in (-\infty, \infty) \times [0,T]$ by the solution $s^i(t)$ of the differential equation

$$\begin{cases} \dfrac{d}{dt} s^i(t) = g^i(s^i(t), P(t)), & t \in [0,T] \\ s^i(t_0) = s_0 \in (-\infty, \infty). \end{cases}$$

For $P \in C([0,T]; \mathbb{R}^N)$, set

$$c^i_P(t) := \varphi^i_P(0, t, 0) = -\int_0^t g^i(0, P(u))\,du\ (\le 0),$$

which is considered as an imaginary initial size of those who are born at time t. Let $z_P^i(t) := \varphi_P^i(t;0,0)$ denote the characteristic curve through $(0,0)$ in the (s,t)-plane. For $(s_0,t_0) \in [c_P^i(T), s_\dagger^i) \times [0,T]$ such that $s_0 < z_P^i(t_0)$, define $\tau_P^i := \tau_P^i(t_0,s_0)$ implicitly by the relation

$$\varphi_P^i(\tau_P^i; t_0, s_0) = 0, \text{ or equivalently, } \varphi_P^i(t_0;\tau_P^i,0) = s_0. \tag{5}$$

For $c \in [c_P^i(T), s_\dagger^i)$, set

$$t_c^i = \begin{cases} \tau_P^i(0,c) & \text{if } c < 0, \\ 0 & \text{if } c \geq 0. \end{cases}$$

We define

$$\begin{aligned} W_P^i(t,u;c) &= \exp\left[-\int_u^t \partial_s g^i(\varphi_P^i(\sigma;0,c), P(\sigma))\,d\sigma\right], \\ U_P^i(t,u;c,p) &= \exp\left[-\int_u^t \mu^i(\varphi_P^i(\sigma;0,c), P_m(\sigma))\,d\sigma\right] \\ \mathcal{U}_P^i(t,u;c,p) &= W_P^i(t,u;c)U_P^i(t,u;c,p) \end{aligned} \tag{6}$$

for $t_c^i \leq u \leq t \leq T$ and $p \in L_T$, where $P_m(t)$ is defined by (2) and depends on p. Let

$$F^i(\phi) = \int_0^{s_\dagger^i} \beta^i(s, P_b\phi)\phi^i(s)\,ds \tag{7}$$

for $\phi \in L_0^1$, where $P_b\phi = (P_b^1\phi, \cdots, P_b^N\phi)$ with $P_b^i\phi = \int_0^{s_\dagger^i} b^i(s)\phi^i(s)\,ds$.

Suppose that $\tilde{p}^i(s,t)$ satisfies $(\tilde{\mathrm{P}})$ in a strict way. Put $\nu_c^i(t) := p^i(\varphi_P^i(t;0,c),t)$ for $t \in [t_c^i, T]$ and $c \in [c_P^i(T), s_\dagger^i)$. Then we have

$$\begin{aligned} \frac{d}{dt}\nu_c^i(t) &= \partial_t\tilde{p}^i(\varphi_P^i(t;0,c),t) + \partial_s\tilde{p}^i(\varphi_P^i(t;0,c),t)\frac{d}{dt}\varphi_P^i(t;0,c) \\ &= \partial_t\tilde{p}^i(\varphi_P^i(t;0,c),t) + \partial_s\tilde{p}^i(\varphi_P^i(t;0,c),t)g^i(\varphi_P^i(t;0,c),P(t)) \\ &= -\left[\mu^i(\varphi_P^i(t;0,c),\tilde{P}_m(t)) + \partial_s^i g^i(\varphi_P^i(t;0,c),P(t))\right]\nu_c^i(t). \end{aligned} \tag{8}$$

The differential equation (8) admits a solution written by

$$\nu_c^i(t) = \mathcal{U}_P^i(t, t_c^i; c, \tilde{p})\nu_c^i(t_c^i).$$

For a.e. $s \in (0, z_P^i(t))$, letting $c := c_P^i(\tau_P^i) = \varphi_P^i(0;t,s) < 0$, we have

$$\nu_c^i(t_c^i) = \tilde{p}^i(\varphi_P^i(t_c^i;0,c),t_c^i) = \tilde{p}^i(0,\tau_P^i(t,s)) = \frac{F^i(\tilde{p}(\cdot,\tau_P^i))}{g^i(0,P(\tau_P^i))},$$

where $\tau_P^i = \tau_P^i(t,s)$ is defined by (5) and F^i is defined by (7). Hence we have

$$\tilde{p}^i(s,t) = \mathcal{U}_P^i(t,\tau_P^i;c_P^i(\tau_P^i),\tilde{p})\frac{F^i(\tilde{p}(\cdot,\tau_P^i))}{g^i(0,P(\tau_P^i))} = \mathcal{U}_P^i(t,\tau_P^i;\varphi_P^i(0;t,s),\tilde{p})\frac{F^i(\tilde{p}(\cdot,\tau_P^i))}{g^i(0,P(\tau_P^i))}$$

for a.e. s $\in (0, z_P^i(t))$. For a.e. $s \in (z_P^i(t), s_\dagger)$, letting $c = \varphi_P^i(0;t,s) > 0$,

$$\nu_c^i(t_c^i) = \tilde{p}^i(\varphi_P^i(0;0,c),0) = p_0^i(\varphi_P^i(0;t,s)).$$

Then we have

$$\tilde{p}^i(s,t) = \mathcal{U}_P^i(t,0;\varphi_P^i(0;t,s),\tilde{p})p_0(\varphi_P^i(0;t,s))$$

for a.e. s $\in (z_P^i(t), s_\dagger^i)$. From above observation, we define a solution of ($\tilde{\text{P}}$) by putting $p_P = \tilde{p}$ as follows.

Definition 2.1. For $P \in C([0,T];\mathbb{R}^N)$, a function $p_P \in L_T$ is said to be a *solution* of ($\tilde{\text{P}}$) if p_P satisfies

$$p_P^i(s,t) = \begin{cases} \mathcal{U}_P^i(t,\tau_P^i;c_P^i(\tau_P^i),p_P)\dfrac{F^i(p_P(\cdot,\tau_P^i))}{g^i(0,P(\tau_P^i))}, & a.e.\ s \in (0, z_P^i(t)) \\ \mathcal{U}_P^i(t,0;\varphi_P^i(0;t,s),p_P)p_0^i(\varphi_P^i(0;t,s)), & a.e.\ s \in (z_P^i(t), s_\dagger^i) \end{cases}$$

where $\tau_P^i = \tau_P^i(t,s)$ is defined by (5) and F^i is defined by (7).

If we can find $P \in C([0,T];\mathbb{R}^N)$ satisfying

$$P^i(t) = \int_0^{s_\dagger^i} w^i(s)p_P^i(s,t)\,ds, \tag{9}$$

$p_P \in L_T$ is certainly a solution of (P) and hence we define a solution of (P) as follows:

Definition 2.2. A function $p \in L_T$ is said to be a *solution* of (P) if $p = p_P$ is a solution of ($\tilde{\text{P}}$) for $P \in C([0,T];\mathbb{R}^N)$ satisfying (9).

Proposition 2.3. *Let* $p \in L_T$ *be a solution of* (P)*. Then we have*

$$\begin{aligned} P^i(t) = &\int_0^t w^i(\varphi_P^i(t;u,0))U_P^i(t,u;c_P^i(u),p)F^i(p(\cdot,u))\,du \\ &+ \int_0^{s_\dagger^i} w^i(\varphi_P^i(t;0,\xi))U_P^i(t,0;\xi,p)p_0^i(\xi)\,d\xi. \end{aligned} \tag{10}$$

Proof. By change of variables $u = \tau_P^i(t,s)$ and $\xi = \varphi_P^i(0;t,s)$, we have

$$\begin{aligned} P^i(t) &= \int_0^{z_P^i(t)} w^i(s)\mathcal{U}_P^i(t,\tau_P^i;c_P^i(\tau_P^i),p)\frac{F^i(p(\cdot,\tau_P^i))}{g^i(0,P(\tau_P^i))}\,ds \\ &\quad + \int_{z_P^i(t)}^{s_\dagger^i} w^i(s)\mathcal{U}_P^i(t,0;\varphi_P^i(0;t,s),p)p_0^i(\varphi_P^i(0;t,s))\,ds \\ &= \int_0^t w^i(\varphi_P^i(t;u,0))U_P^i(t,u;c_P^i(u),p)F^i(p(\cdot,u))\,du \\ &\quad + \int_0^{s_\dagger^i} w^i(\varphi_P^i(t;0,\xi))U_P^i(t,0;\xi,p)p_0^i(\xi)\,d\xi. \end{aligned}$$

Thus (10) holds. □

Our main result is stated as follows:

Theorem 2.4. *Let* (H1)–(H4) *hold. Then for any initial value* $p_0 \in L^1_{0,+}$*, there exists a unique solution* $p \in C([0,\infty); L^1_{0,+})$ *of* (P) *satisfying the following estimate:*

$$\|p(\cdot,t)\|_{L^1} \le e^{\bar{\beta}t}\|p_0\|_{L^1}, \quad t \in [0,\infty). \tag{11}$$

3. Lemmas

In this section, we prepare some lemmas to prove Theorems 2.4. Throughout this section, we assume (H1)–(H4). First, we recall the following Gronwall's lemma:

Lemma 3.1 (Gronwall's Lemma)**.** *Let* $c \in C[0,T]$*,* $c(t) \ge 0$ *and* $f \in C^1[0,T]$*. Let* $a \in [0,T]$ *be fixed.*

(i) *If* $v \in C[0,T]$ *satisfies*

$$v(t) \le f(t) + \int_a^t c(s)v(s)\,ds, \quad t \in [a,T]. \tag{12}$$

Then we have

$$\begin{aligned} v(t) &\le f(t) + \int_a^t \exp\left(\int_s^t c(\tau)\,d\tau\right) c(s)f(s)\,ds \\ &= \exp\left(\int_a^t c(\tau)\,d\tau\right) f(a) + \int_a^t \exp\left(\int_s^t c(\tau)\,d\tau\right) f'(s)\,ds, \quad t \in [a,T]. \end{aligned} \tag{13}$$

(ii) *If* $v \in C[0,T]$ *satisfies*

$$v(t) \le f(t) + \int_t^a c(s)v(s)\,ds \quad t \in [0,a]. \tag{14}$$

Then we have

$$\begin{aligned} v(t) &\le f(t) + \int_t^a \exp\left(\int_t^s c(\tau)\,d\tau\right) c(s)f(s)\,ds \\ &= \exp\left(\int_t^a c(\tau)\,d\tau\right) f(a) - \int_t^a \exp\left(\int_t^s c(\tau)\,d\tau\right) f'(s)\,ds, \quad t \in [0,a]. \end{aligned} \tag{15}$$

Proof. That (12) implies (13) follows from usual Gronwall's lemma and the integration by parts. To show that (14) implies (15), put

$$q(t) := \int_t^a c(s)v(s)\,ds, \quad t \in [0,a].$$

Then q is of class C^1 and satisfies $q'(t) = -c(t)v(t)$ for $t \in (0,a)$. By (14) and the positivity of $c(t)$, we have

$$q'(t) \ge -c(t)f(t) - c(t)q(t), \quad t \in (0,a).$$

Then

$$\begin{aligned} &\frac{d}{dt}\left\{q(t)\exp\left(-\int_t^a c(\tau)\,d\tau\right)\right\} = [q'(t) + c(t)q(t)]\exp\left(-\int_t^a c(\tau)\,d\tau\right) \\ &\ge -c(t)f(t)\exp\left(-\int_t^a c(\tau)\,d\tau\right), \quad t \in (0,a). \end{aligned}$$

Integrating the above inequality over $[t, a]$, we obtain

$$q(a) - q(t)\exp\left(-\int_t^a c(\tau)\,d\tau\right) \geq -\int_t^a c(s)f(s)\exp\left(-\int_s^a c(\tau)\,d\tau\right)\,ds.$$

Since $q(a) = 0$,

$$q(t) \leq \int_t^a c(s)f(s)\exp\left(\int_t^s c(\tau)\,d\tau\right)\,ds, \quad t \in [0, a].$$

Then by (14) and the integration by parts, we conclude that (15) holds. □

Lemma 3.2. *Let P, $\hat{P} \in C([0,T];\mathbb{R}^N)$ and $\|P\|_{C([0,T];\mathbb{R}^N)}$, $\|\hat{P}\|_{C([0,T];\mathbb{R}^N)} \leq r$. Then we have*

$$|\varphi_P^i(t;t_0,s_0) - \varphi_{\hat{P}}^i(t;t_0,s_0)| \leq c_g(r)e^{c_g(r)T}\left|\int_{t_0}^t |P(\eta) - \hat{P}(\eta)|_N\,d\eta\right|, \tag{16}$$

where $c_g(r)$ appears in (H3).

Proof. By definition of characteristic curves and (H3),

$$\begin{aligned}
&|\varphi_P^i(t;t_0,s_0) - \varphi_{\hat{P}}^i(t;t_0,s_0)| \\
&\leq \left|\int_{t_0}^t |g^i(\varphi_P^i(\sigma;t_0,s_0),P(\sigma)) - g^i(\varphi_{\hat{P}}^i(\sigma;t_0,s_0),\hat{P}(\sigma))|\,d\sigma\right| \\
&\leq \left|\int_{t_0}^t c_g(r)\left(|\varphi_P^i(\sigma;t_0,s_0) - \varphi_{\hat{P}}^i(\sigma;t_0,s_0)| + |P(\sigma) - \hat{P}(\sigma)|_N\right)\,d\sigma\right|.
\end{aligned}$$

For $t \geq t_0$, we have

$$\begin{aligned}
&|\varphi_P^i(t;t_0,s_0) - \varphi_{\hat{P}}^i(t;t_0,s_0)| \\
&\leq \int_{t_0}^t c_g(r)|\varphi_P^i(\sigma;t_0,s_0) - \varphi_{\hat{P}}^i(\sigma;t_0,s_0)|\,d\sigma + \int_{t_0}^t c_g(r)|P(\sigma) - \hat{P}(\sigma)|_N\,d\sigma.
\end{aligned}$$

For $t < t_0$, we have

$$\begin{aligned}
&|\varphi_P^i(t;t_0,s_0) - \varphi_{\hat{P}}^i(t;t_0,s_0)| \\
&\leq \int_t^{t_0} c_g(r)|\varphi_P^i(\sigma;t_0,s_0) - \varphi_{\hat{P}}^i(\sigma;t_0,s_0)|\,d\sigma + \int_t^{t_0} c_g(r)|P(\sigma) - \hat{P}(\sigma)|_N\,d\sigma.
\end{aligned}$$

Then Lemma 3.1 implies (16). □

Lemma 3.3. *Let P, $\hat{P} \in C([0,T];\mathbb{R}^N)$ and let p_P, $p_{\hat{P}} \in L_{T,+}$ be the corresponding solutions to* $(\tilde{\mathrm{P}})$ *with initial values p_0, $\hat{p}_0 \in L_{0,+}$ satisfying $\|p_P\|_{L_T}$, $\|p_{\hat{P}}\|_{L_T} \leq r$. Then for $0 \leq \eta \leq u \leq t \leq T$, $\xi \in [0, s_\dagger^i)$, we have the following estimate:*

$$\begin{aligned}
&|U_P^i(t,u;\varphi_P^i(0;\eta,\xi),p_P) - U_{\hat{P}}^i(t,u;\varphi_{\hat{P}}^i(0;\eta,\xi),p_{\hat{P}})| \\
&\leq \Gamma_1(r,T)\int_0^t \left(|P_m(\sigma) - \hat{P}_m(\sigma)|_N + |P(\sigma) - \hat{P}(\sigma)|_N\right)\,d\sigma
\end{aligned} \tag{17}$$

where $\Gamma_1(r,T)$ is a constant depending on r and T.

Proof. It follows from (6) and the mean value theorem that

$$\begin{aligned}
&|U^i_P(t,u;\varphi^i_P(0;\eta,\xi),p_P) - U^i_{\hat P}(t,u;\varphi^i_{\hat P}(0;\eta,\xi),p_{\hat P})| \\
&\le \int_u^t |\mu^i(\varphi^i_P(\sigma,\eta,\xi),P_m(\sigma)) - \mu^i(\varphi^i_{\hat P}(\sigma,\eta,\xi),\hat P_m(\sigma))|\,d\sigma.
\end{aligned}$$

By (H1) and Lemma 3.2,

$$\begin{aligned}
&|\mu^i(\varphi^i_P(\sigma,\eta,\xi),P_m(\sigma)) - \mu^i(\varphi^i_{\hat P}(\sigma,\eta,\xi),\hat P_m(\sigma))| \\
&\le c_\mu(\bar m r)\left(|\varphi^i_P(\sigma,\eta,\xi) - \varphi^i_{\hat P}(\sigma,\eta,\xi)| + |P_m(\sigma) - \hat P_m(\sigma)|_N\right) \\
&\le c_\mu(\bar m r)\left(c_g(r)e^{c_g(r)T}\int_\eta^\sigma |P(\sigma) - \hat P(\sigma)|_N\,d\sigma + |P_m(\sigma) - \hat P_m(\sigma)|_N\right).
\end{aligned}$$

Then, we have (17) with $\Gamma_1(r,T) := c_\mu(\bar m r)\left(c_g(r)\mathrm{e}^{c_g(r)T} + 1\right)$. □

Lemma 3.4. *Let P, $\hat P \in C([0,T];\mathbb{R}^N)$. Let p_P, $p_{\hat P} \in L_{T,+}$ be the solutions of $(\tilde{\mathrm{P}})$ with initial values p_0, $\hat p_0 \in L_{0,+}$ and suppose that $\|p_P\|_{L_T}$, $\|p_{\hat P}\|_{L_T} \le r$. Then we have*

$$\begin{aligned}
&|F^i(p_P(\cdot,t)) - F^i(p_{\hat P}(\cdot,t))| \\
&\le \Gamma_2(r,T)\left(|P_b(t) - \hat P_b(t)|_N + \int_0^t |P_b(\tau) - \hat P_b(\tau)|_N\,d\tau \right.\\
&\quad \left. + \int_0^t |P_m(\tau) - \hat P_m(\tau)|_N\,d\tau + \int_0^t |P(\tau) - \hat P(\tau)|_N\,d\tau\right) \\
&\quad + \Gamma_3(r,T)\|p_0 - \hat p_0\|_{L^1},
\end{aligned} \tag{18}$$

where $\Gamma_2(r,T)$ and $\Gamma_3(r,T)$ are some constants depending on r, T.

Proof. Note first that by (H2), if $\|p_P\|_{L_T} \le r$, the following estimate holds:

$$|F^i(p_P(\cdot,t))| \le \int_0^{s^i_\dagger} |\beta^i(s,t,P_b(t))p^i_P(s,t)|\,ds \le \bar\beta r. \tag{19}$$

Similarly to Proposition 2.3, we have

$$\begin{aligned}
F^i(p_P(\cdot,t)) = &\int_0^t \beta^i(\varphi^i_P(t;u,0),P_b(t))U^i_P(t,u;c^i_P(u),p_P)F^i(p_P(\cdot,u))\,du \\
&+ \int_0^{s^i_\dagger} \beta^i(\varphi^i_P(t;0,\xi),P_b(t))U^i_P(t,0;\xi,p_P)p^i_0(\xi)\,d\xi.
\end{aligned} \tag{20}$$

It follows from (20) that

$$
\begin{aligned}
&|F^i(p_P(\cdot,t)) - F^i(p_{\hat{P}}(\cdot,t))| \\
&\le \int_0^t |\beta^i(\varphi^i_P(t;u,0), P_b(t)) U^i_P(t,u;c^i_P(u),p_P) F^i(p_P(\cdot,u)) \\
&\qquad - \beta^i(\varphi^i_{\hat{P}}(t;u,0), \hat{P}_b(t)) U^i_{\hat{P}}(t,u;c^i_{\hat{P}}(u),\hat{p}_{\hat{P}}) F^i(p_{\hat{P}}(\cdot,u))|\, du \\
&\quad + \int_0^{s^i_\dagger} |\beta^i(\varphi^i_P(t;0,\xi), P_b(t)) U^i_P(t,0;\xi,p_P) p^i_0(\xi) \\
&\qquad - \beta^i(\varphi^i_{\hat{P}}(t;0,\xi), \hat{P}_b(t)) U^i_{\hat{P}}(t,0;\xi,p_{\hat{P}}) \hat{p}^i_0(\xi)|\, d\xi =: K_1 + K_2.
\end{aligned}
$$

By (H2) and (19),

$$
\begin{aligned}
K_1 &\le \bar{\beta} r c_\beta(\bar{b}r) \int_0^t \Big(|\varphi^i_P(t;u,0) - \varphi^i_{\hat{P}}(t;u,0)| + |P_b(t) - \hat{P}_b(t)|_N \Big)\, du \\
&\quad + \bar{\beta}^2 r \int_0^t |U^i_P(t,u;c^i_P(u),p_P) - U^i_{\hat{P}}(t,u;c^i_{\hat{P}}(u),p_{\hat{P}})|\, du \\
&\quad + \bar{\beta} \int_0^t |F^i(p_P(\cdot,u)) - F^i(p_{\hat{P}}(\cdot,u))|\, du \\
K_2 &\le c_\beta(\bar{b}r) \int_0^{s^i_\dagger} \Big(|\varphi^i_P(t;0,\xi) - \varphi^i_{\hat{P}}(t;0,\xi)| + |P_b(t) - \hat{P}_b(t)|_N \Big) |p^i_0(\xi)|\, d\xi \\
&\quad + \bar{\beta} \int_0^{s^i_\dagger} |U^i_P(t,0;\xi,p_P) - U^i_{\hat{P}}(t,0;\xi,p_{\hat{P}})||p^i_0(\xi)|\, d\xi + \bar{\beta} \int_0^{s^i_\dagger} |p^i_0(\xi) - \hat{p}^i_0(\xi)|\, d\xi.
\end{aligned}
$$

By Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
K_1 &\le \bar{\beta} r c_\beta(\bar{b}r) \int_0^t \left(c_g(r) e^{c_g(r)T} \int_u^t |P(\eta) - \hat{P}(\eta)|_N\, d\eta + |P_b(t) - \hat{P}_b(t)|_N \right) du \\
&\quad + \bar{\beta}^2 r \int_0^t \left(\Gamma_1(r,T) \int_0^t \Big(|P_m(\sigma) - \hat{P}_m(\sigma)|_N + |P(\sigma) - \hat{P}(\sigma)|_N \Big)\, d\sigma \right. \\
&\quad \left. + \Gamma_2(r,T) \int_0^u |P(\tau) - \hat{P}(\tau)|_N\, d\tau \right) du + \bar{\beta} \int_0^t |F^i(p_P(\cdot,u)) - F^i(p_{\hat{P}}(\cdot,u))|\, du, \\
K_2 &\le c_\beta(\bar{b}r) \left(c_g(r) e^{c_g(r)T} \int_0^t |P(\eta) - \hat{P}(\eta)|_N\, d\eta + |P_b(t) - \hat{P}_b(t)|_N \right) \|p_0\|_{L^1} \\
&\quad + \bar{\beta} \left(\Gamma_1(r,T) \int_0^t \Big(|P_m(\sigma) - \hat{P}_m(\sigma)|_N + |P(\sigma) - \hat{P}(\sigma)|_N \Big)\, d\sigma \right) \|p_0\|_{L^1} \\
&\quad + \bar{\beta} \|p_0 - \hat{p}_0\|_{L^1}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&|F^i(p_P(\cdot,t)) - F^i(p_{\hat{P}}(\cdot,t))| \\
&\le \bar{\beta} \int_0^t |F^i(p_P(\cdot,u)) - F^i(p_{\hat{P}}(\cdot,u))|\, du \\
&\quad + C(r,T) \left(|P_b(t) - \hat{P}_b(t)|_N + \int_0^t \left(|P_m(\sigma) - \hat{P}_m(\sigma)|_N + |P(\sigma) - \hat{P}(\sigma)|_N \right) d\sigma \right) \\
&\quad + \bar{\beta} \|p_0 - \hat{p}_0\|_{L^1},
\end{aligned}
$$

where $C(r,T)$ is a constant depending on r and T. Then by Gronwall's lemma, the desired estimate (18) holds. □

4. Proof of Theorem 2.4

Define a closed subset E of $C([0,T];\mathbb{R}^N)$ by

$$
E := \left\{ P \in C([0,T];\mathbb{R}^N_+) \;\middle|\; P^i(0) = \int_0^{s^i_\dagger} w^i(s) p^i_0(s)\, ds \right\}.
$$

Step 1. Given $P \in E$, put $\tilde{g}^i_P(s,t) := g^i(s, P(t))$. Problem $(\tilde{\mathrm{P}})$ can be written in the following form:

$$
(21) \qquad \begin{cases} \partial_t \tilde{p}^i + \partial_s(\tilde{g}^i_P(s,t)\tilde{p}^i) = G^i(\tilde{p}(\cdot,t))(s) & s \in [0, s^i_\dagger),\ t \in [0,T], \\ \tilde{g}^i_P(0,t)\tilde{p}^i(0,t) = F^i(\tilde{p}(\cdot,t)) & t \in [0,T], \\ \tilde{p}^i(s,0) = p^i_0(s), & s \in [0, s^i_\dagger), \end{cases}
$$

where F^i is defined by (7) and G^i is defined by

$$
G^i(\phi)(s) = -\mu^i(s, P_m\phi)\phi^i(s),\ \ a.e.\ s \in (0, s^i_\dagger)
$$

for $\phi \in L^1_0$, where $P_m\phi$ is defined similarly to $P_b\phi$ appearing in (7). Let $F(\phi) = (F^1(\phi), \cdots, F^N(\phi))$ and $G(\phi)(s) = (G^1(\phi)(s), \cdots, G^N(\phi)(s))$. It is shown that $F : L^1_0 \to \mathbb{R}^N$, $G : L^1_0 \to L^1_0$, and there exist increasing functions $c_F,\ c_G : [0,\infty) \to [0,\infty)$ such that

$$
|F(\phi_1) - F(\phi_2)|_N \le c_F(r)\|\phi_1 - \phi_2\|_{L^1}, \quad \|G(\phi_1) - G(\phi_2)\|_{L^1} \le c_G(r)\|\phi_1 - \phi_2\|_{L^1}
$$

for $\phi_1,\ \phi_2 \in L^1_0$. It is obvious that $F(\phi) \in \mathbb{R}^N_+$ for $\phi \in L^1_{0,+}$ and $G(\phi) + \bar{\mu}\phi \in L^1_{0,+}$ for $\phi \in L^1_{0,+}$. Furthermore,

$$
\sum_{i=1}^N \left[F^i(\phi) + \int_0^{s^i_\dagger} G^i(\phi)(s)\, ds \right] \le \bar{\beta}\|\phi\|_{L^1}
$$

for $\phi \in L^1_{0,+}$. Then we can apply the results of [5] and problem (21), and hence $(\tilde{\mathrm{P}})$ admits a unique global solution $p_P \in L_{T,+}$ such that

$$
p^i_P(s,t) = \begin{cases} \mathcal{U}^i_P(t, \tau^i_P; c^i_P(\tau^i_P), p_P) \dfrac{F^i(p_P(\cdot, \tau^i_P))}{\tilde{g}^i_P(0, \tau^i_P)}, & a.e.\ s \in (0, z^i_P(t)) \\ \mathcal{U}^i_P(t, 0; \varphi^i_P(0; t, s), p_P) p^i_0(\varphi^i_P(0; t, s)), & a.e.\ s \in (z^i_P(t), s^i_\dagger) \end{cases}
$$

where $\tau_P^i = \tau_P^i(t,s)$, and p_P satisfies

$$\|p_P(\cdot,t)\|_{L^1} \le e^{\bar\beta t}\|p_0\|_{L^1}. \tag{22}$$

Step 2. Let $[K_wP](t) = ([K_wP]^1(t), \cdots, [K_wP]^N(t))$ with

$$[K_wP]^i(t) := \int_0^{s_\dagger^i} w^i(s)p_P^i(s,t)\,ds,\ \ t\in[0,T]. \tag{23}$$

It is obvious that K_w maps E into itself. Our aim is to find a fixed point $P\in E$ of K_w by using Banach's fixed point theorem. Then since $P^i(t) = \int_0^{s_\dagger^i} w^i(s)p_P^i(s,t)\,ds$, it is evident that p_P corresponding to the fixed point P becomes the solution of (P). In order to treat different nonlocal terms, we introduce auxiliary mappings K_b and K_m on E similarly to K_w as follows:

$$[K_bP]^i(t) := \int_0^{s_\dagger^i} b^i(s)p_P^i(s,t)\,ds,\ \ t\in[0,T],$$
$$[K_mP]^i(t) := \int_0^{s_\dagger^i} m^i(s)p_P^i(s,t)\,ds,\ \ t\in[0,T]$$

for $P\in E$. Recall that $[K_wP]^i$ defined by (23) is represented by the right hand of (10), that is,

$$\begin{aligned}[K_wP]^i(t) = &\int_0^t w^i(\varphi_P^i(t;u,0))U_P^i(t,u;c_P^i(u),p_P)F^i(p_P(\cdot,u))\,du\\ &+\int_0^{s_\dagger^i} w^i(\varphi_P^i(t;0,\xi))U_P^i(t,0;\xi,p_P)p_0^i(\xi)\,d\xi.\end{aligned} \tag{24}$$

Letting $r := e^{\bar\beta T}\|p_0\|_{L^1}$, we have $\|p_P(\cdot,t)\|_{L^1}\le r$ by (22) and then $|F^i(p_P(\cdot,u))| \le \bar\beta r$. Let $P,\ \hat P\in E$. It follows from (24) that

$$\begin{aligned}\Big|[K_wP]^i(t) - [K_w\hat P]^i(t)\Big| &\le \bar w\int_0^t |F^i(p_P(\cdot,u)) - F^i(p_{\hat P}(\cdot,u))|\,du\\
&+\bar\beta r\int_0^t |w^i(\varphi_P^i(t;u,0)) - w^i(\varphi_{\hat P}^i(t;u,0))|\,du\\
&+\bar w\bar\beta r\int_0^t |U_P^i(t,u;c_P^i(u),p_P) - U_{\hat P}^i(t,u;c_{\hat P}^i(u),p_{\hat P})|\,du\\
&+\int_0^{s_\dagger^i} |w^i(\varphi_P^i(t;0,\xi)) - w^i(\varphi_{\hat P}^i(t;0,\xi))||p_0^i(\xi)|\,d\xi\\
&+\bar w\int_0^{s_\dagger^i} |U_P^i(t,0;\xi,p_P) - U_{\hat P}^i(t,0;\xi,p_{\hat P})||p_0^i(\xi)|\,d\xi.\end{aligned}$$

Using Lemmas 3.2–3.4, we have

$$\begin{aligned}|K_wP(t) - K_w\hat P(t)|_N \le \bar w\tilde\Gamma(r,T)\int_0^t \Big(&|P(u)-\hat P(u)|_N\\ &+|K_bP(u)-K_b\hat P(u)|_N + |K_mP(u)-K_m\hat P(u)|_N\Big)\,du\end{aligned} \tag{25}$$

for some constant $\tilde{\Gamma}(r,T) > 0$ depending on r and T. Similarly, we have

$$
\begin{aligned}
|K_b P(t) - K_b \hat{P}(t)|_N \le \bar{b}\tilde{\Gamma}(r,T) \int_0^t \Big(&|P(u) - \hat{P}(u)|_N \\
&+ |K_b P(u) - K_b \hat{P}(u)|_N + |K_m P(u) - K_m \hat{P}(u)|_N \Big)\, du,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
|K_m P(t) - K_m \hat{P}(t)|_N \le \bar{m}\tilde{\Gamma}(r,T) \int_0^t \Big(&|P(u) - \hat{P}(u)|_N \\
&+ |K_b P(u) - K_b \hat{P}(u)|_N + |K_m P(u) - K_m \hat{P}(u)|_N \Big)\, du.
\end{aligned} \tag{27}
$$

Put

$$
\Psi(t) := |K_b P(t) - K_b \hat{P}(t)|_N + |K_m P(t) - K_m \hat{P}(t)|_N
$$

and $\omega := \bar{w} + \bar{b} + \bar{m}$. Then it follows from (25)–(27) that

$$
\begin{aligned}
&|K_w P(t) - K_w \hat{P}(t)|_N + \Psi(t) \\
&\le \omega\tilde{\Gamma}(r,T) \int_0^t \Psi(u) du + \omega\tilde{\Gamma}(r,T) \int_0^t |P(u) - \hat{P}(u)|_N \, du
\end{aligned} \tag{28}
$$

It is easily seen that

$$
\Psi(t) \le \omega\tilde{\Gamma}(r,T) \int_0^t \Psi(u) du + \omega\tilde{\Gamma}(r,T) \int_0^t |P(u) - \hat{P}(u)|_N \, du.
$$

By Gronwall's lemma, we have

$$
\Psi(t) \le \omega\tilde{\Gamma}(r,T) \mathrm{e}^{\omega\tilde{\Gamma}(r,T)} \int_0^t |P(\sigma) - \hat{P}(\sigma)|_N \, d\sigma. \tag{29}
$$

It follows from (28) and (29) that

$$
|K_w P(t) - K_w \hat{P}(t)|_N \le C(r,T) \int_0^t |P(u) - \hat{P}(u)|_N \, du \tag{30}
$$

for some constant $C(r,T) > 0$. We introduce a norm on $C([0,T];\mathbb{R}^N)$, which is equivalent to the usual norm by

$$
\|P\|_\lambda := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} |P(t)|_N \quad \text{for } P \in C([0,T];\mathbb{R}^N),
$$

where $\lambda > 0$ is determined later. Then it follows from (30) that

$$
\|K_w P - K_w \hat{P}\|_\lambda \le \sup_{t\in[0,T]} e^{-\lambda t} C(r,T) \int_0^t |P(u) - \hat{P}(u)|_N \, du \le \frac{C(r,T)}{\lambda} \|P - \hat{P}\|_\lambda
$$

for P, $\hat{P} \in E$. Therefore, choosing $\lambda > C(r,T)$, K_w becomes a contraction on E. Finally, note that (11) holds from (22). This completes the proof.

References

[1] A. S. Ackleh, H. T. Banks, and K. Deng, A finite difference approximation for a coupled system of nonlinear size-structured populations, Nonlinear Anal. 50, (2002) 727–748.

[2] A. Calsina and J. Saldaña, A model of physiologically structured population dynamics with a nonlinear individual growth rate, J. Math. Biol. 33, (1995) 335–364.

[3] M. E. Gurtin and R. C. MacCamy, Non-linear age-dependent population dynamics, Arch. Rational. Mech. Anal. 54, (1974) 281–300.

[4] N. Kato, A general model of size-dependent population dynamics with nonlinear growth rate, J. Math. Anal. Appl. 297, (2004) 234–256.

[5] N. Kato, A system of non-autonomous nonlinear size-structured population dynamics, Special Issue of Dynamics of Continuous, Discrete and Impulsive Systems Series A 14(S2) (2007) 79–84.

[6] G. F. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, New York, 1985.

Kosuke Kamizaki: Department of Mathematics, Shimane University, Matsue, Shimane, 690-8504, Japan

Nobuyuki Kato: Department of Mathematics, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: nkato@riko.shimane-u.ac.jp