島根大学教育学部紀要（自然科学）第26巻　1頁～5頁　平成4年12月

A New Proof of a Conjecture of Good

Dedicated to Professor Masanori Kishi on his 60th birthday

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Introduction

Let $N \geqq 2$ be a fixed integer. A real number x $(0 \leqq x \leqq 1)$ is expressed as decimal in the scale N (i.e., involving digits $0, 1, \ldots, N-1$), and let $P(x; q, r)$ denote the number of times the digit r occurs among the first q digits of its decimal.

In [3; p.200] I. J. Good raised the following

PROBLEM. Let $\{p_r\}_{r=0}^{N-1}$ be a sequence of nonnegative numbers such that $\Sigma_{r=0}^{N-1} p_r = 1$, and set $\alpha = -\Sigma_{r=0}^{N-1} p_r \log p_r / \log N$. If S is the set of x $(0 \leqq x \leqq 1)$ for which

$$\lim_{q \to \infty} P(x; q, r)/q = p_r \qquad (r = 0, \cdots, N-1),$$

then is it true that the fracitional dimension of the set S is equal to α?

In [2] H. G. Eggleston proved that this is true, and P. Billingsley obtained some more general results on a regular Markov chain ([1]). In particular, as a byproduct he proved

THEOREM ([1; THEOREM 7.1]). *Let S^* be the set of x $(0 \leqq x \leqq 1)$ for which*

$$P(x; q, r)/q = p_r + O(1/q) \text{ as } q \to \infty, \text{ for } r = 0, \cdots, N-1.$$

Then the set S^* also has the same fractional dimension α, where $\{p_r\}_{r=0}^{N-1}$ and α are given above.

In this note, we shall constructively show that $\dim S^* \geqq \alpha$ and so $\dim S^* = \alpha$, because S^*

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This research was paritially supported by Grand-in-Aid for Scientific Research (NO. 03640159), Ministry of Education.

$\subset S$ and dim $S \leqq \alpha$ (this is relatively easily obtained), where dim A denotes the fractional dimension of a set A.

1. Lemmas

To prove the above theorem we prepare three lemmas. The last one is used to obtain the lower estimation of the fracitonal dimension of S^*.

LEMMA 1. *Let* $0 \leqq a \leqq 1$. *Then there exists a sequence* $\{\varepsilon_j\}_{j=1}^{\infty}$ *such that* ε_j *is* 0 *or* 1 *and*

$$0 \leqq a - k^{-1}\Sigma_{j=1}^{k} \varepsilon_j < k^{-1} \quad \textit{for } k \geqq 1. \tag{1}$$

PROOF. We determine the integer ε_j inductively. First, take $\varepsilon_1 = 0$, so it satisfies (1) for $k = 1$. Suppose $\varepsilon_1, \cdots, \varepsilon_k$ are obtained and they satisfy inequalities (1). Since the half-open interval $((k+1)a - \Sigma_{j=1}^{k}\varepsilon_j - 1, (k+1)a - \Sigma_{j=1}^{k}\varepsilon_j]$ contains a unique integer, we take it as ε_{k+1}. Then inequalities (1) replaced k by $k+1$ are fulfilled with $\varepsilon_1, \cdots, \varepsilon_{k+1}$. It remains to prove that ε_{k+1} is 0 or 1. By the choice of it and the assumption of the induction we have

$$-1 \leqq -1 + a < \varepsilon_{k+1} < 1 + a < 2,$$

which implies the desired result.

REMARK. By replacing a by $1-a$ in the lemma, we can prove that for given a, $0 < a \leqq 1$, there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that ε_j is 0 or 1 and

$$0 \leqq k^{-1}\Sigma_{j=1}^{k}\varepsilon_j - a < k^{-1} \quad \textit{for } k \geqq 1.$$

Let h be an increasing continuous function defined on $[0, \infty)$ with $h(0) = 0$. We denote Λ_h(A) the Haudorff measure of a set A. In case $h(r) = r^{\alpha}$ for $\alpha > 0$, we write Λ_α instead of Λ_h.

For the sake of completeness, we quote [4; Lemma 1] as Lemma 2, but omit its proof.

LEMMA 2. *Let F be a closed set in the interval* $[0, 1]$ *and let* $\mathfrak{A}$ *be the family of open sets* $(\neq \phi)$ *in R each of which is a finite union of open intervals. Assume that there exists a nonnegative set function* Φ *on* $\mathfrak{A}$ *satisfying the following conditions:*

(i) *if* $\omega = \cup_{i=1}^{k}\omega_i$, $\omega_i \in \mathfrak{A}$ $(i = 1, \cdots, k)$, *then* $\Phi(\omega) \leqq \Sigma_{i=1}^{k}\Phi(\omega_i)$,

(ii) *if* $\omega \in \mathfrak{A}$ *contains F, then* $\Phi(\omega) \geqq b$, *where b is some positive constant,*

(iii) *there exist positive constants a and* d_0 *such that if I is any interval with length* $|I|\leqq d_0$, *then* $\Phi(I)\leqq ah(|I|)$.

Then $\Lambda_h(F)\geqq b/a$.

Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of integers with $n_j\geqq 1$ for $j\geqq 1$ and let $\{(t_j^{(0)}, \cdots, t_j^{(N-1)})\}_{j=1}^{\infty}$ be a sequence of vectors with positive integral components such that $\Sigma_{r=0}^{N-1} t_j^{(r)}=n_j$ for $j\geqq 1$.

LEMMA 3. *Let h,* $\{n_j\}_{j=1}^{\infty}$ *and* $\{(t_j^{(0)}, \cdots, t_j^{(N-1)})\}_{j=1}^{\infty}$ *be given above. Let E be the set of x* $(0\leqq x\leqq 1)$ *for which*

$$P(x;\ n_1,\ r)=t_1^{(r)}.$$

$$P(x;\ n_1+\cdots+n_j,\ r)-P(x;\ n_1+\cdots+n_{j-1},\ r)=t_j^{(r)} \text{ for } j\geqq 2 \text{ and } r=0, \cdots, N-1.$$

Suppose the sequence $\{n_j\}_{j=1}^{\infty}$ *is bounded. Then the set E is closed and*

$$M^{-1} \liminf_{j\to\infty} K_1\cdots K_j h(\ell_j) \leqq \Lambda_h(E) \leqq \liminf_{j\to\infty} K_1\cdots K_j h(\ell_j), \tag{2}$$

where $K_j=n_j!/(t_j^{(0)}!\cdots t_j^{(N-1)}!)$, $\ell_j=N^{-n_1-\cdots-n_j}$ *for* $j\geqq 1$ *and* $M=2\max_{j\geqq 1} K_j$.

REMARK. This is simpler than that of [1; Theorem 4.3], but our assumptions are more restrictive than that of the same theorem.

PROOF. It is clear that E is closed (this is true without the boundedness assumption of $\{n_j\}_{j=1}^{\infty}$). Thus in the sequel we prove the inequalities (2). Since E is covered by a union of $K_1\cdots K_j$ closed intervals with length ℓ_j, the upper estimate is easily obtained. Hence it suffices to prove the lower estimate.

To see this, we may assume that $\liminf_{j\to\infty} K_1\cdots K_j h(\ell_j)>0$. Let $0<b<\liminf_{j\to\infty} K_1\cdots K_j h(\ell_j)$. Thus there exists an integer j_0 such that $b<K_1\cdots K_j h(\ell_j)$ for $j\geqq j_0$. Taking a sequence $\{\lambda_j\}_{j=j_0}^{\infty}$ of positive numbers such that $b=K_1\cdots K_j h(\lambda_j)$, then $0<\lambda_j<\ell_j$ for $j\geqq j_0$. So we define the set function Φ as follows:

$$\Phi(\omega)=\lim_{j\to\infty} N_j(\omega)h(\lambda_j)$$

for ω is an open set, where $N_j(\omega)$ is the number of intervals of type $[a_1N^{-1}+\cdots+a_{n_1+\cdots+n_j}N^{-n_1-\cdots-n_j},\ a_1N^{-1}+\cdots+(a_{n_1+\cdots+n_j}+1)N^{-n_1-\cdots-n_j}]$ which meet ω. Here the number of the elements of the set $\{k; n_1+\cdots+n_{i-1}<k\leqq n_1+\ldots+n_i,\ a_k=r\}$ is equal to $t_i^{(r)}$ for i, $1\leqq i\leqq j$ and r, $0\leqq r\leqq N-1$. We note that the right side limit exists, because the sequence $N_j(\omega)h(\lambda_j)$ is decreasing. It is easily checked that Φ satisfies conditions (i) and

(ii) of Lemma 2. Thus it remains only to prove that it satisfies (iii) of the lemma.

Let I be an open interval with length $|I|$ less than ℓ_{j_0}. Then there exists an integer $j \geqq j_0$ such that $\ell_{j+1} \leqq |I| < \ell_j$, so $N_{j+1}(I) \leqq 2K_{j+1} \leqq M$. Thus we have

$$\Phi(I) \leqq N_{j+1}(I) h(\lambda_{j+1}) \leqq Mh(|I|),$$

since $\lambda_{j+1} \leqq \ell_{j+1} \leqq |I|$. Therefore the condition (iii) with $a=M$ and $d_0=\ell_{j_0}$ is satisfied. It follws from Lemma 2 that $\Lambda_h(E) \geqq M^{-1}b$. Since b is an arbitrary number such that $b < \liminf_{j\to\infty} K_1 \cdots K_j h(\ell_j)$, the desired estimate is obtained.

Question. In this lemma, if we drop the assumption that $\{n_j\}_{j=1}^{\infty}$ is bounded, is the assertion still true for some positive constant M ?

2. Proof of the theorem

In this section we construct a sequence $\{E_n\}$ of subsets of S^* such that $\liminf_{n\to\infty} \dim E_n \geqq \alpha$ and from this $\dim S^* \geqq \alpha$. To do so, we may assume that $0 < p_0 \leqq \cdots \leqq p_{N-1} < 1$, since in case some $p_r = 1$, then $\alpha = 0$, so the assertion is clear, and in case some of p_r are zero, by modifying the following proof, the conclusion can be obtained.

Let n_0 be an integer such that $Nn_0p_0 \geqq 1$. For $n \geqq n_0$ *and* $r=0, \cdots, N-2$, we put $m_r = [Nnp_r]$ and $m_{N-1} = Nn - \Sigma_{r=0}^{N-2} m_r$, so $1 \leqq m_r \leqq Nnp_r < m_r + 1 (0 \leqq r \leqq N-2)$. By Lemma 1 there exist sequences $\{\varepsilon_j^{(r)}\}_{j=1}^{\infty}$ such that $\varepsilon_j^{(r)} = 0$ or 1, and for $k \geqq 1$ and r, $0 \leqq r \leqq N-2$

$$0 \leqq Nnp_r - m_r - k^{-1}\Sigma_{j=1}^{k} \varepsilon_j^{(r)} < k^{-1}.$$

Taking $n_j = Nn$, $t_j^{(r)} = m_r + \varepsilon_j^{(r)} (r=0, \cdots, N-2)$ and $t_j^{(N-1)} = n_j - \Sigma_{r=0}^{N-2} t_j^{(r)}$ for $j \geqq 1$, the set E defined as in Lemma 3 is denoted by E_n. Then $E_n \subset S^*$. In order to prove this, let $x \in E_n$ and $q > Nn$. Then there exists an integer $k(\geqq 1)$ such that $kNn \leqq q < (k+1)Nn$. For $r=0 \cdots, N-2$, we have by the choice of the sequences $\{\varepsilon_j^{(r)}\}$

$$km_r + \Sigma_{j=1}^{k} \varepsilon_j^{(r)} \leqq P(x; q, r) \leqq (k+1)m_r + \Sigma_{j=1}^{k+1} \varepsilon_j^{(r)},$$

so

$$P(x; q, r)/q \leqq \{m_r + k^{-1}\Sigma_{j=1}^{k} \varepsilon_j^{(r)}\}/(Nn) + (m_r+1)/(kNn) \leqq p_r + 2(m_r+1)/q.$$

Similarly, we have

$$P(x; q, r)/q \geqq p_r - (m_r+2)/q.$$

Thus we obtain for $r=0, \cdots, N-2$,

(3) $$P(x;\, q,\, r)/q = p_r + O(1/q) \qquad \text{as } q\to\infty.$$

Since $\Sigma_{r=0}^{N-1} P(x;\, q,\, r) = q$, from (3) it follows that

$$P(x;\, q,\, N-1)/q = p_{N-1} + O(1/q) \qquad \text{as } q\to\infty.$$

Thus we have proved that $E_n \subset S^*$. It remains to prove that $\liminf_{j\to\infty} \dim E_n \geqq \alpha$.

To see this, put $\min_j\{n_j!/(t_j^{(0)}!\cdots t_j^{(N-1)}!)\} = (Nn)!/(s_0!\cdots s_{N-1}!)$. Then $s_r = m_r + \delta_r$ for $r=0, \cdots, N-2$, $s_{N-1} = Nn - \Sigma_{r=0}^{N-2} s_r$, where $\delta_r = 0$ or 1. Let $\beta_n = (Nn\log N)^{-1}\log\{(Nn)!/(s_0!\cdots s_{N-1}!)\}$. Then by Lemma 3 we obtain $\Lambda_{\beta_n}(E_n) > 0$ and thus $\dim E_n \geqq \beta_n$, since

$$\{\Pi_{i=1}^{j} n_i!/(t_i^{(0)}!\cdots t_i^{(N-1)}!)\} \times N^{-jNn\beta_n} \geqq \{(Nn)!/(s_0!\cdots s_{N-1}!) \times N^{-Nn\beta_n}\}^j = 1.$$

By Stirling's formula, it can be shown that $\lim_{n\to\infty} \beta_n = \alpha$, which completes the proof.

References

[1] P. Billingsley, Hausdorff dimensions in probability theory, Ill, J. Math. **41**(1961), 187 – 209.

[2] H. G. Eggleston, The fractional dimension of a set defined by decimal properties, Quart. J. Oxford, **20** (1949), 31 – 36.

[3] I. J. Good, The fractional dimensional theory of continued fractions, Proc. Cambridge Philos. Soc. **37** (1941), 199 – 228.

[4] K. Hatano, Evaluation of Hausdorff measures generalized Cantor sets, Sci. Hiroshima Univ. Ser. A – I. **32**(1968), 371 – 379.