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A note on (α, p)-thinness of symmetric generalized Cantor sets

Dedicated to Professor M. Yamada on the ocassion of his 60th birthday

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1. Introduction. Let g_α be a Bessel kernel of order α, $0<\alpha<\infty$, on the n-dimensional Euclidean space $R^n(n \geqq 1)$, whose Fourier transform is $(1+|\xi|^2)^{-\alpha/2}$. The Bessel capacity $B_{\alpha,p}$ is defined as follows: For a set $A \subset R^n$,

$$B_{\alpha,p}(A) = \inf \int f(x)^p dx,$$

where the infimum is taken over all functions $f \in L_p^+$ such that

$$g_\alpha * f(x) \geqq 1 \quad \text{for all } x \in A.$$

We shall always assume that $1<p<\infty$ and $0<\alpha p \leqq n$. We say that a set A is (α,p)-thin at $x \in R^n$ (see, [5]) if

$$\int_0^1 \{ r^{\alpha p-n} B_{\alpha,p}(A \cap B(x, r)) \}^{1/(p-1)} r^{-1} dr < \infty,$$

where $B(x, r)$ denotes the open ball with center at x and radius r.

In [4; Theorem 2] Hedberg and Wolff have proved that the Kellogg property, i.e., $B_{\alpha,p}(A \cap e(A)) = 0$ for any set $A \subset R^n$, where $e(A) = \{x \in R^n; A \text{ is } (\alpha,p)\text{-thin at } x\}$, also holds in the non-linear potential theory. It is easily seen from this property that $B_{\alpha,p}(A) = 0$ if and only if A is (α,p)-thin at all of its points. In this note in a special case where E is a symmetric generalized Cantor set (for the definition, see [3]), we prove the following

THEOREM. *Let E be the symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^\infty, \{\ell_j\}_{j=0}^\infty]$ with $\ell_0 < 1$. Then the following three assertions are mutually equivalent:*

(a) $B_{\alpha,p}(E) = 0$;

(b) E is (α,p)-thin at some point $x \in E$;

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(c) $\sum_{j=1}^{\infty} u_j v_j = \infty$,

where $u_j = (k_1 \cdots k_j)^{-n/(p-1)}$ and

$$v_j = \begin{cases} \ell_j^{(\alpha p - n)/(p-1)} & \text{if } \alpha p < n, \\ \max\{-\log \ell_j, 1\} & \text{if } \alpha p = n. \end{cases}$$

2. Proof of the theorem. To prove the theorem we prepare two lemmas. We owe the proof of Lemma 1 to professor F-Y. Maeda.

LEMMA 1. *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of positive numbers satisfying the following conditions:*

(a) *There is a positive number $\lambda < 1$ such that $a_{i+1} < \lambda a_i$ for all i;*

(b) *$\{b_i\}$ is monotone increasing and $b_i \to \infty$ $(i \to \infty)$.*

If $\sum a_i b_i < \infty$, then

$$\sum_{i=2}^{\infty} a_i (b_i - b_{i-1}) (\sum_{j=i}^{\infty} a_j b_j)^{-1} = \infty.$$

PROOF. (i) The case $\liminf_{i\to\infty} b_{i-1} b_i^{-1} < 1$. In this case, we find a positive number $\mu < 1$ and a sequence of positive integers $\{n_k\}$ such that $n_k \to \infty$ $(k \to \infty)$ and $b_{n_k - 1} < \mu b_{n_k}$ for all k. Note that $b_{n_k - 1} < \mu (1 - \mu)^{-1} (b_{n_k} - b_{n_k - 1})$ for all k. Since

$$\sum_{j=i}^{\infty} a_j (b_j - b_{j-1}) = \sum_{j=i}^{\infty} a_j b_j - \sum_{j=i-1}^{\infty} a_{j+1} b_j$$
$$= \sum_{j=i}^{\infty} (a_j - a_{j+1}) b_j - a_i b_{i-1} \geqq (1 - \lambda) \sum_{j=i}^{\infty} a_j b_j - a_i b_{i-1},$$

we have

$$\sum_{j=n_k}^{\infty} a_j b_j \leqq (1 - \lambda)^{-1} \{ \sum_{j=n_k}^{\infty} a_j (b_j - b_{j-1}) + a_{n_k} b_{n_k - 1} \}$$
$$\leqq (1 - \lambda)^{-1} \{ \sum_{j=n_k}^{\infty} a_j (b_j - b_{j-1}) + \mu (1 - \mu)^{-1} a_{n_k} (b_{n_k} - b_{n_k - 1}) \}$$
$$\leqq (1 - \lambda)^{-1} (1 - \mu)^{-1} \sum_{j=n_k}^{\infty} a_j (b_j - b_{j-1}).$$

Hence,

$$\sum_{i=n_k}^{\infty} a_i (b_i - b_{i-1}) (\sum_{j=i}^{\infty} a_j b_j)^{-1}$$
$$\geqq \sum_{i=n_k}^{\infty} a_i (b_i - b_{i-1}) (\sum_{j=n_k}^{\infty} a_j b_j)^{-1} \geqq (1 - \lambda)(1 - \mu) > 0$$

for all k. Thus, we have the desired result.

(ii) The case $\lim_{i\to\infty} b_{i-1} b_i^{-1} = 1$. In this case, there is i_0 such that $b_{i-1} b_i^{-1} > \lambda^{1/2}$ for all $i \geqq i_0$.

Hence,

$$\Sigma_{j=i}^{\infty} a_j b_j \leqq a_i b_i \Sigma_{m=0}^{\infty} \lambda^{m/2} = M^{-1} a_i b_i$$

for all $i \geqq i_0$, where $M = 1 - \lambda^{1/2}$. Therefore, for any $k \geqq i_0$,

$$\Sigma_{i=k}^{\infty} a_i (b_i - b_{i-1}) (\Sigma_{j=i}^{\infty} a_j b_j)^{-1} \geqq M \Sigma_{i=k}^{\infty} a_i (b_i - b_{i-1}) a_i^{-1} b_i^{-1}$$

$$\geqq M \lim_{m\to\infty} b_m^{-1} \Sigma_{i=k}^{m} (b_i - b_{i-1}) = M \lim_{m\to\infty} b_m^{-1} (b_m - b_{k-1}) = M,$$

which implies the desired result.

LEMMA 2 ([3; THEOREM]). *Let E be the symmetric generalized Cantor set in R^n constructed by the system* $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$ *with* $\ell_0 < 1$. *Then there is a constant* $C > 1$ *dependent only on n, p and α such that*

$$C^{-1} (v_0 + \Sigma_{j=1}^{\infty} u_j v_j)^{1-p} \leqq B_{\alpha,p}(E) \leqq C (\Sigma_{j=1}^{\infty} u_j v_j)^{1-p}.$$

PROOF OF THE THEOREM. The implication (c)=>(a) follows from Lemma 2, and the implication (a)=>(b) is trivial by the definition of the (α, p)-thinness.

(b)=>(c): It suffices to show that if $\Sigma_{j=1}^{\infty} u_j v_j < \infty$, then

$$\int_0^1 \{r^{\alpha p - n} B_{\alpha,p}(E \cap B(x, r))\}^{1/(p-1)} r^{-1} dr = \infty$$

for any $x \in E$. Let i_0 be an integer $\geqq 3$ such that $2^{i_0 - 1} > n^{1/2}$. Then $n^{1/2} \ell_{i-1} < 1$ for $i \geqq i_0$. Also, note that $-\log \ell_i > 1$ for $i \geqq 2$. Given $x \in E$, for each $i \geqq i_0$, x is contained an n-dimensional cube $I_n^{(i)}$ of length ℓ_i which appears in the definition of the Cantor set E. Then $I_n^{(i)} \subset B(x, \ell_i')$, so that

$$B_{\alpha,p}(E \cap I_n^{(i)}) \leqq B_{\alpha,p}(E \cap B(x, \ell_i')),$$

where $\ell_i' = n^{1/2} \ell_i$. Since $E \cap I_n^{(i)}$ is a symmetric generalized Cantor set constructed by the system $[\{k_{i+j}\}_{j=1}^{\infty}, \{\ell_{i+j}\}_{j=0}^{\infty}]$, by Lemma 2 we obtain

$$B_{\alpha,p}(E \cap I_n^{(i)}) \geqq C^{-1} (v_i + \Sigma_{j=i+1}^{\infty} u_j u_i^{-1} v_j)^{1-p}$$

$$= C^{-1} u_i^{p-1} (\Sigma_{j=i}^{\infty} u_j v_j)^{1-p}.$$

Hence,

$$\int_0^1 \{r^{\alpha p-n}B_{\alpha,p}(E\cap B(x,\ r))\}^{1/(p-1)}r^{-1}dr$$
$$\geqq \Sigma_{i=i_0}^{\infty}\int_{\ell'_i}^{\ell'_{i-1}} \{r^{\alpha p-n}B_{\alpha,p}(E\cap B(x,\ r))\}^{1/(p-1)}r^{-1}dr$$
$$\geqq \Sigma_{i=i_0}^{\infty} B_{\alpha,p}(E\cap B(x,\ell'_i))^{1/(p-1)}\int_{\ell'_i}^{\ell'_{i-1}} r^{(\alpha p-n)/(p-1)-1}dr$$
$$\geqq C'\Sigma_{i=i_0}^{\infty} u_i\,(v_i-v_{i-1})\,(\Sigma_{j=i}^{\infty} u_jv_j)^{-1}$$

with a positive constant C'. If $\Sigma\, u_jv_j<\infty$, then Lemma 1 shows that the last expression in the above inequalities is ∞. Thus the implication (b)$=>$(c) is proved.

REMARK. The (α,p)-fine topology $\tau_{\alpha,p}$ is defined by the family
$\{H\subset R^n\ ;\ R^n\setminus H$ is (α,p)-thin at every point of $H\}$.
In [3], we constructed a symmetric generalized Cantor set E such that

$$(*)\quad (R^n\setminus E)\cup\{x^0\}\in\tau_{\beta,q}\setminus\tau_{\alpha,p}\ \text{for}\ x^0\in E,$$

in the following four cases: (i) $0<\beta q<\alpha p<n$, (ii) $0<\beta q<\alpha p=n$, (iii) $0<\beta q=\alpha p<n$ and $q>p$ and (iv) $0<\beta q=\alpha p=n$ and $q>p$. The above theorem shows that we can not obtain a symmetric generalized Cantor set E satifying ($*$) in the remainder case, namely in case $0<\alpha p\leqq\beta q<n$ and $(n-\alpha p)/(p-1)<(n-\beta q)/(q-1)$ (cf. [1 ; Theorem B]). In fact, if there is such a set E, then E is (β,q)-thin at x^0, so that $B_{\beta,q}(E)=0$ by the theorem. But this implies that $B_{\alpha,p}(E)=0$, since $\alpha p<\beta q$ or $\alpha p=\beta q$ and $p>q$ (see, [2 ; Theorem 5 . 5]); and hence E is (α,p)-thin at x^0, which contradicts ($*$).

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