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A Remark on Fractional Dimensions of Difference Sets

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Introduction

Under the continuum hypothesis W. Sierpiński [7] proved that a set E which possesses 'the property C' is of measure zero with respect to any Hausdorff measure but $E-E=R^1$. In his proof we can see that a difference set $A-B$ is closely related to the orthogonal projection of the product set $A\times B$ in the xy-plane to the line $y=-x$. In [8] D. J. Ward defined an n-difference set $D^n(E)$ of a non empty set $E\subset R^1$ and showed that dim $D(E)\leqq\min\{n\alpha,\ n-1\}$ under the conditions that the set E is an α-set and it has positive lower density with respect to the α-dimensional Hausdorff measure at every point in it.

In this remark we shall estimate the lower and upper bounds of fractional dimensions of difference sets and show that the upper bound is sharp.

In §1, following [4] we shall define a perfect set of translation and under some condition we shall evaluate the Hausdorff measure of it in §2. In §3 we shall discuss the fractional dimensions of difference sets.

§1. Definitions

Let R^n $(n\geqq 1)$ be the n-dimensional Euclidean space with points $x=(x_1,\ x_2,\dots,\ x_n)$. By an n-dimensional open cube (resp. closed cube) in R^n, we mean the set of points $x=(x_1,\ x_2,\ \dots,\ x_n)$ satisfying the inequalities :

$a_i<x_i<a_i+d$ (resp. $a_i<x_i\leqq a_i+d$) for $i=1,\ 2,\dots,\ n$, where a_i $(i=1,\ 2,\dots,\ n)$ are any numbers and $d>0$. We call d the length of the side, or simply the side, of the open (or closed) cube.

Let $\mathfrak{A}$ be the family of non empty open sets in R^n which is determined by the following properties :

(i) any n-dimensional open cube belongs to $\mathfrak{A}$,

(ii) if ω_1 and ω_2 belong to $\mathfrak{A}$, then so does $\omega_1\cup\omega_2$,

(iii) if ω is an element of $\mathfrak{A}$, then there exist a finite number of n-dimensional open cubes I_ν $(\nu=1,\ 2,\dots,\ N)$ such that $\omega=\bigcup_{\nu=1}^{N} I_\nu$.

Let E be a subset in R^n. For any number α, $0<\alpha<n$, we define the α-dimensional Hausdorff outer measure of E by

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$$\Lambda_\alpha(E)=\lim_{\rho\to+0}\{\inf \sum d_\nu^\alpha\},$$

where the infimum is taken over all coverings of E by at most a countable number of open cubes with sides $d_\nu \leqq \rho$.

The fractional dimension dim E of a set E is defined by

$$\dim E=\inf\{\alpha\ ;\ \Lambda_\alpha(E)=0\}.$$

A set E is an α-set if it is measurable with respect to the α-dimensional Hausdorff outer measure and $0<\Lambda_\alpha(E)<\infty$. Note that the fractional dimension of an α-set is α.

We denote by $c(x, r)$ the closed ball in R^n with center x and radius r.

The upper and lower circular densities of a set E in R^n at a point x are defined by

$$\overline{D}^{(\alpha)}(E, x)=\overline{\lim_{r\to 0}}\ 2^{-\alpha}r^{-\alpha}\Lambda_\alpha(E\cap c(x, r)) \text{ and}$$

$$\underline{D}^{(\alpha)}(E, x)=\underline{\lim}_{r\to 0}\ 2^{-\alpha}r^{-\alpha}\Lambda_\alpha(E\cap c(x, r)).$$

Let θ be a number such that $0\leqq\theta<\pi$. We denote by $\text{proj}_\theta E$ the linear set obtained by the orthogonal projection of a plane set E onto the line $y\cos(\theta+\pi/2)=x\sin(\theta+\pi/2)$.

Let A and B be non empty subsets in R^1. The difference set A and B is defined by the set of all numbers $a-b$ with $a\in A$ and $b\in B$, denoted by $A-B$. Then we can see that

(1) $\text{proj}_{\pi/4}(A\times B)=\{((a-b)/2,\ -(a-b)/2)\ ;\ a\in A,\ b\in B\}$ and

(2) $\Lambda_\alpha(A-B)=2^{\alpha/2}\Lambda_\alpha(\text{proj}_{\pi/4}(A\times B))$ for any positive α.

From (1) it follows easily that the difference set of Cantor's ternary set is equal to $[-1, 1]$. In the following the equality (2) will be often used.

We shall define a perfect set of translation (cf. [4]). Let $\nu\geqq 2$ be an integer and $\eta_1, \eta_2, \ldots, \eta_\nu$ be numbers such that

$$0\leqq\eta_1<\eta_2<\ldots<\eta_\nu<1.$$

Let ξ be a number satisfying the following inequalities:

$$0<\xi,\ \xi<\eta_2-\eta_1,\ \xi<\eta_3-\eta_2,\ldots,\ \xi<\eta_\nu-\eta_{\nu-1},\ \xi\leqq 1-\eta_\nu.$$

We remove intervals from a closed interval $[a, b]$ $(a<b)$ so that ν disjoint

closed intervals $[a+(b-a)\eta_j,\ a+(b-a)(\eta_j+\xi)]$ $(j=1, 2,\dots, \nu)$ remain. We call this operation on the closed interval $[a, b]$ the dissection of type $(\nu, \eta_1, \eta_2,\dots, \eta_\nu, \xi)$.

Let $\{\nu_q\}_{q=1}^{\infty}$ be a sequence of integers, $\{(\eta_{q,1}, \eta_{q,2},\dots, \eta_{q,\nu_q})\}_{q=1}^{\infty}$ be a sequence of sets of numbers and $\{\xi_q\}_{q=1}^{\infty}$ be a sequence of positive numbers. Suppose a system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2},\dots,\eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ satisfies the following condition :

$$(3)\begin{cases} \nu_q\geqq 2,\ 0\leqq \eta_{q,1}<\eta_{q,2}<\dots<\eta_{q,\nu_q}<1 \text{ and} \\ 0<\xi_q,\ \xi_q<\eta_{q,2}-\eta_{q,1},\dots,\ \xi_q<\eta_{q,\nu_q}-\eta_{q,\nu_q-1}, \\ \xi_q\leqq 1-\eta_{q,\nu_q}\ (q\geqq 1). \end{cases}$$

At the first step, we operate the dissection of type $(\nu_1, \eta_{1,1}, \eta_{1,2},\dots, \eta_{1,\nu_1}, \xi_1)$ on the interval $[0, 1]$ and obtain ν_1 closed intervals with side ξ_1. Next we operate the dissection of type $(\nu_2, \eta_{2,1}, \eta_{2,2},\dots,\eta_{2,\nu_2}, \xi_2)$ on each interval and obtain $\nu_1\nu_2$ closed intervals with length $\xi_1\xi_2$. We continue this process. At the qth step, we have $\nu_1\nu_2\dots\nu_q$ closed intervals with length $\xi_1\xi_2\dots\xi_q$. Let E_q be the union of these intervals. We define $E^{(1)}=\bigcap_{q=1}^{\infty}E_q$. It is called the one-dimensional perfect set of translation constructed by the system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2}, \dots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$. We call the product set $E^{(n)}=E^{(1)}\times E^{(1)}\times\dots\times E^{(1)}$ of n one-dimensional perfect set of translation of $E^{(1)}$ the n-dimensional set of translation constructed by the system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2}, \dots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$. We can see that $E^{(n)}=\bigcap_{q=1}^{\infty}E_q\times E_q\times\dots\times E_q$, where $E_q\times E_q\times\dots\times E_q$ is a product set in R^n and consists of $(\nu_1\nu_2\dots\nu_q)^n$ n-dimensional cubes with side $\xi_1\xi_2\dots\xi_q$. We call $E_q\times E_q\times\dots\times E_q$ the qth approximation of $E^{(n)}$ $(n\geqq 1)$.

If $\nu_q=\nu$, $\eta_{q,1}=\eta_1$, $\eta_{q,2}=\eta_2,\dots, \eta_{q,\nu_q}=\eta_\nu$ and $\xi_q=\xi$ $(q\geqq 1)$, then we call the n-dimensional perfect set of translation constructed by the system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2}, \dots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ the n-dimensional homogenous perfect set of type $(\nu, \eta_1, \eta_2,\dots,\eta_\nu, \xi)$.

If $\eta_{q,1}=0$, $\eta_{q,2}=(\lambda_q+\delta_q)\lambda_{q-1}^{-1}$, $\eta_{q,2}=2(\lambda_q+\delta_q)\lambda_{q-1}^{-1},\dots, \eta_{q,\nu_q}=(\nu_q-1)(\lambda_q+\delta_q)\lambda_{q-1}^{-1}$ and $\xi_q=\lambda_{q-1}^{-1}\lambda_q$ $(q\geqq 1)$, where $\lambda_0=1$, then it is easy to see that the n-dimensional perfect set of translation constructed by the system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2},\dots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ is equal to the n-dimensional symmetric generalized Cantor set constructed by the system $[l=1, \{\nu_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=1}^{\infty}]$. We refer to [2] for the definition of symmetric generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=1}^{\infty}]$. In the following $l=1$ will be omitted.

§2. Evaluation of Hausdorff measures of perfect sets of translation

LEMMA 1. ([6]) *Let F be a compact set in R^n and let $\mathfrak{A}$ be the family defined in* §1. *Assume that there exists a non-negative set function Φ on $\mathfrak{A}$ satisfying the following conditions:*

(1) *if* $\omega=\bigcup_{i=1}^{N}\omega_i$, $\omega_i\in\mathfrak{A}$ $(i=1, 2,\ldots, N)$, *then* $\Phi(\omega)\leqq\sum_{i=1}^{N}\Phi(\omega_i)$,

(2) *if* $\omega\in\mathfrak{A}$ *contains* F, *then* $\Phi(\omega)\geqq b$, *where b is some positive constant,*

(3) *there exist positive constants a and d_0 such that if I is any n-dimensional open cube with side $d\leqq d_0$, then $\Phi(I)\leqq ad^{\alpha}$.*

Then $\Lambda_\alpha(F)\geqq b/a$.

Given a system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1},\eta_{q,2},\ldots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ satisfying (3) in §1, we write

$$\delta_{q+1}=(\nu_{q+1}-1)^{-1}(\xi_1\xi_2\ldots\xi_q-\nu_{q+1}\xi_1\xi_2\ldots\xi_{q+1}),$$
$$\delta_{q+1,1}=\xi_1\xi_2\ldots\xi_q\eta_{q+1,1},$$
$$\delta_{q+1,2}=\xi_1\xi_2\ldots\xi_q(\eta_{q+1,2}-\eta_{q+1,1}-\xi_{q+1}),$$
$$\ldots,$$
$$\delta_{q+1,\nu_{q+1}}=\xi_1\xi_2\ldots\xi_q(\eta_{q+1,\nu_{q+1}}-\eta_{q+1,\nu_{q+1}-1}-\xi_{q+1})$$

and

$$\delta_{q+1,\nu_{q+1}+1}=\xi_1\xi_2\ldots\xi_q(1-\eta_{q+1,\nu_{q+1}}-\xi_{q+1})\ (q\geqq 0),$$

where $\xi_0=1$. Let m_q be the number of integers j which satisfy $\delta_{q,j}<\delta_q$, provided the set of such integers j is not empty, and $m_q=0$ if it is empty $(q\geqq 1)$. By a method similar to that in the proof of Theorem 1 in [3] we obtain

THEOREM 1. *Let $E^{(n)}$ be n-dimensional perfect set of translation constructed by the system* $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2},\ldots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ *satisfying condition* (3) *in* §1. *If the sequence $\{m_q\}_{q=1}^{\infty}$ is bounded, then we have*

$$a^{-1}\varliminf_{q\to\infty}(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^{\alpha}\leqq\Lambda_\alpha(E^{(n)})\leqq\varliminf_{q\to\infty}(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^{\alpha}\ (0<\alpha<n),$$

where a is a positive constant.

PROOF. From the definition of the Hausdorff outer measure the right-hand inequality is obvious. Therefore it is sufficient to prove the left-hand inequality in the case $A=\varliminf_{q\to\infty}(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^{\alpha}>0$. Let B be an arbitrary number such that $0<B<A$. Then there exists a positive integer q_0 such that $(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^{\alpha}>B$ for all $q\geqq q_0$. We choose sequences $\{\mu_q\}_{q=q_0}^{\infty}$ and $\{\delta'_q\}_{q=q_0+1}^{\infty}$ such that $(\nu_1\nu_2\ldots\nu_q)^n\mu_q^{\alpha}=B$ for $q\geqq q_0$ and $\nu_{q+1}\mu_{q+1}+(\nu_{q+1}-1)\delta'_{q+1}=\mu_q$ for $q\geqq q_0$. Clearly $\mu_q<\xi_1\xi_2\ldots\xi_q$ for $q\geqq q_0$. It is easy to see that $\{N_q(\omega)\mu_q^{\alpha}\}_{q=q_0}^{\infty}$ is a decreasing sequence for every $\omega\in\mathfrak{A}$, where $N_q(\omega)$ is the number of n-dimensional closed cubes in the qth approximation of $E^{(n)}$ which meet ω.

Now we define a set function Φ on $\mathfrak{A}$ by $\Phi(\omega)=\lim_{q\to\infty} N_q(\omega)\mu_q^\alpha$. We can easily check that Φ satisfies conditions (1) and (2) of Lemma 1 with $F=E^{(n)}$ and $b=B$. We set $d_0=\xi_1\xi_2\ldots\xi_{q_0+1}$ and $a=2^{2n}(M+3)^n+2^{4n}$, where $M=\sup_{q\geqq 1} m_q$. Let I be any open cube with side $d\leqq d_0$. Then there exist uniquely determined positive integers q $(\geqq q_0+1)$ and j $(1\leqq j\leqq \nu_{q+1}-1)$ such that $\xi_1\xi_2\ldots\xi_{q+1}<d\leqq\xi_1\xi_2\ldots\xi_q$ and $j\xi_1\xi_2\ldots\xi_{q+1}+(j-1)\delta_{q+1}<d\leqq(j+1)\xi_1\xi_2\ldots\xi_{q+1}+j\delta_{q+1}$.

Since $E^{(n)}$ is symmetric, we have

$$N_{q+1}(I)\leqq\{2(M+j+2)\}^n.$$

If $j=1$, then

$$\Phi(I)\leqq\{2(M+3)\}^n\mu_{q+1}^\alpha<a(\xi_1\xi_2\ldots\xi_{q+1})^\alpha<ad^\alpha.$$

If $2\leqq j\leqq\nu_{q+1}-1$, then $(j^{n/\alpha}-j)/(j-1)\leqq(\nu_{q+1}^{n/\alpha}-\nu_{q+1})/(\nu_{q+1}-1)=\delta'_{q+1}/\mu_{q+1}$. It follows that $j^n\mu_{q+1}^\alpha\leqq\{j\mu_{q+1}+(j-1)\delta'_{q+1}\}^\alpha$. On the other hand,

$$\begin{aligned}
&j\mu_{q+1}+(j-1)\delta'_{q+1}<j(\mu_{q+1}+\delta'_{q+1})\\
&\leqq 2jB^{1/\alpha}(\xi_1\xi_2\ldots\xi_q)^{-n/\alpha}\nu_{q+1}^{-1}\\
&=2j\nu_{q+1}^{-1}\mu_q<2j\nu_{q+1}^{-1}\xi_1\xi_2\ldots\xi_q\\
&<2j(\xi_1\xi_2\ldots\xi_{q+1}+\delta_{q+1})\\
&<4\{j\xi_1\xi_2\ldots\xi_{q+1}+(j-1)\delta_{q+1}\}<4d.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Phi(I)\leqq\{2(M+j+2)\}^n\mu_{q+1}^\alpha<2^{2n}\{(M+2)^n+j^n\}\mu_{q+1}^\alpha\\
&\leqq\{2^{2n}(M+3)^n+2^{4n}\}d^\alpha=ad^\alpha.
\end{aligned}$$

Thus Φ satisfies condition (3) in Lemma 1. By Lemma 1 we obtain $\Lambda_\alpha(E^{(n)})\geqq B/a$. Since B is an arbitrary number less than A, we have

$$a^{-1}\varliminf_{q\to\infty}(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^\alpha\leqq\Lambda_\alpha(E^{(n)}).$$

This is the desired inequality.

Counting $N_{q+1}(I)$ more precisely in the above proof, we obtain

COROLLARY 1. *Let $E^{(n)}$ be the n-dimensional homogeneous perfect set of type $(\nu, \eta_1, \eta_2,\ldots,\eta_\nu, \xi)$ and take $\alpha=-n\log\nu/\log\xi$. Then $(2\nu)^{-n}\leqq\Lambda_\alpha(E^{(n)})\leqq 1$. Thus the set $E^{(n)}$ is an α-set.*

COROLLARY 2. *Let $E^{(n)}$ be the n-dimensional generalized Cantor set constructed by* *the system* $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=1}^{\infty}]$. *Then we have*

$$2^{-4n} \varliminf_{q\to\infty} (k_1k_2\ldots k_q)^n\lambda_q^{\alpha} \leqq \Lambda_\alpha(E^{(n)}) \leqq \varliminf_{q\to\infty} (k_1k_2\ldots k_q)^n\lambda_q^{\alpha} \quad (0<\alpha<n).$$

COROLLARY 3. *Under the condition of the theorem, we obtain*

$$\dim E^{(n)} = \varliminf_{q\to\infty} \log (\nu_1\nu_2\ldots\nu_q)^n/-\log(\xi_1\xi_2\ldots\xi_q).$$

PROOF. For brevity we set $\alpha_0 = \varliminf_{q\to\infty} \log (\nu_1\nu_2\ldots\nu_q)^n/-\log(\xi_1\xi_2\ldots\xi_q)$. If $\beta > \alpha_0$, then there exists a subsequnce $\{q_j\}_{j=1}^{\infty}$ such that $\beta > \log(\nu_1\nu_2\ldots\nu_{q_j})^n/-\log(\xi_1\xi_2\ldots\xi_{q_j})$ for every $j \geqq 1$. Thus $1 > (\nu_1\nu_2\ldots\nu_{q_j})^n(\xi_1\xi_2\ldots\xi_{q_j})^\beta$ for every j. It follows from the theorem that $\Lambda_\beta(E^{(n)}) \leqq 1$ and hence $\dim E^{(n)} \leqq \beta$. Since β is an arbitrary number such that $\beta > \alpha_0$, we have $\dim E^{(n)} \leqq \alpha_0$.

On the other hand, if $0 < \alpha < \alpha_0$, then there exists a positive integer q_0 such that $\alpha < \log(\nu_1\nu_2\ldots\nu_q)^n/-\log(\xi_1\xi_2\ldots\xi_q)$ for all $q \geqq q_0$. Thus $1 < (\nu_1\nu_2\ldots\nu_q)^n (\xi_1\xi_2\ldots\xi_q)^\alpha$ for all $q \geqq q_0$. It follows from the theorem that $\Lambda_\alpha(E^{(n)}) > 0$ and hence $\dim E^{(n)} \geqq \alpha$. It implies that $\dim E^{(n)} \geqq \alpha_0$. Hence we have the corollary.

If the sequence $\{m_q\}_{q=1}^{\infty}$ is not bounded, we can not obtain the result of the theorem.

EXAMPLE. Let α be a number such that $0<\alpha<n$. Let $E^{(n)}$ be the n-dimensional perfect set of translation constructed by the system $[\{\nu_q\}_{q=1}^{\infty}, \{(\eta_{q,1}, \eta_{q,2}, \ldots, \eta_{q,\nu_q})\}_{q=1}^{\infty}, \{\xi_q\}_{q=1}^{\infty}]$ which satisfies $\lim_{q\to\infty} \nu_q = \infty$, $\xi_q = \nu_q^{-n/\alpha}$, $(2\nu_q+1)\,\xi_q < 1$ and $\eta_{q,1}=0$, $\eta_{q,2}=2\xi_q$, $\eta_{q,3}=4\xi_q, \ldots, \eta_{q,\nu_q} = 2\nu_q\xi_q$ $(q \geqq 1)$. Then $E^{(n)}$ is covered by $(\nu_1\nu_2\ldots\nu_{q-1})^n$ closed cubes with side $(2\nu_q+1)(\xi_1\xi_2\ldots\xi_q)$. Thus $\Lambda_\alpha(E^{(n)}) \leqq \varliminf_{q\to\infty} (\nu_1\nu_2\ldots\nu_{q-1})^n\{(2\nu_q+1)\ \xi_1\xi_2\ldots\xi_q\}^\alpha = 2^\alpha \lim_{q\to\infty} \nu_q^{\alpha-n} = 0$. However it is easy to check that $(\nu_1\nu_2\ldots\nu_q)^n(\xi_1\xi_2\ldots\xi_q)^\alpha = 1$ and $m_q = \nu_q$ $(q \geqq 1)$.

§3. Hausdorff dimension of difference sets

LEMMA 2. ([1], [5]) *Let α and β be positive numbers such that $0<\alpha, \beta<1$. Let A and B be subsets in R^1 such that $0<\Lambda_\alpha(A)<\infty$ and $0<\Lambda_\beta(B)<\infty$. If at every point of B the lower circular density is positive, then the fractional dimension of the product set $A\times B$ is $\alpha+\beta$ and $0<\Lambda_{\alpha+\beta}(A\times B)$.*

LEMMA 3. *Let $E^{(n)}$ be the n-dimensional homogeneous perfect set of type $(\nu, \eta_1, \ldots, \eta_\nu, \xi)$ and take $\alpha = -n \log \nu/\log \xi$. Then*

$\Lambda_\alpha(E^{(n)} \cap c(x, r)) \geqq n^{-\alpha/2}(2\nu)^{-n}\xi^\alpha r^\alpha$ *for all x in $E^{(n)}$ and all r, $0<r<1$.*

PROOF. If x in $E^{(n)}$, then there exists a decreasing sequence $\{I_q\}_{q=1}^{\infty}$ of closed intervals such that $\bigcap_{q=1}^{\infty} I_q = \{x\}$, I_q is contained in the qth approximation of $E^{(n)}$ and the side of I_q is ξ^q. For any number r, $0<r<1$, we choose the smallest integer q_0 such that $c(x, r)\supset I_{q_0}$. It follows that $r<n^{1/2}\xi_0^{q-1}$. So we have, using Corollary 1 to Theorem 1,

$$\Lambda_\alpha(E^{(n)}\cap c(x, r))/r^\alpha \geqq n^{-\alpha/2}\Lambda_\alpha(E^{(n)}\cap I_{q_0})/\xi^{(q_0-1)\alpha} = n^{-\alpha/2}\Lambda_\alpha(E^{(n)})\nu^{-nq_0}\xi^{-(q_0-1)\alpha}$$
$$\geqq n^{-\alpha/2}(2\nu)^{-n}\xi^\alpha.$$

Hence Lemma 3 is proved.

Remark. By modifying the above proof we have : Let E be the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=1}^\infty]$ which satisfies $(k_1k_2\ldots k_q)^n\lambda_q^\alpha=1$ $(q\geqq 1)$. If the sequnece $\{k_q\}_{q=1}^\infty$ is bounded, then $\underline{D}^{(\alpha)}(E, x)>0$ for all x in E. If it is not bounded, then $\underline{D}^{(\alpha)}(E, x)=0$ for all x in E.

THEOREM 2. *Let A and B be non empty subsets in R^1 such that* dim $A=\alpha$, dim $B=\beta$ *and* dim $A\times B=\alpha+\beta$, $0<\alpha, \beta<1$.
Then

$$\max\{\alpha, \beta\}\leqq \dim(A-B)\leqq \min\{1, \alpha+\beta\}$$

and the upper bound can be attained.

PROOF. Since $\Lambda_\gamma(A-B)=2^{\gamma/2}\Lambda_\gamma(\mathrm{proj}_{\pi/4}(A\times B))$ for any positive number γ, it is easily seen that dim $(A-B)\leqq \min\{1, \alpha+\beta\}$.

In case $\beta=(\log k/\log n)\alpha$, $\alpha+\beta<1$ for some integers n and k, $2\leqq n\leqq k$, we shall construct sets attain the upper bound. Set $r=(1-kn^{1-1/\alpha})/(nk-1)$. Then the number r is positive.

Let $\eta_1, \eta_2, \ldots, \eta_n, \eta_1', \eta_2', \ldots, \eta_k', \xi$ and ξ' be numbers such that $\eta_j=(jk-1)(n^{-1/\alpha}+r)$ $(j=1, 2, \ldots, n)$, $\xi=n^{-1/\alpha}$, $\xi_j'=(j-1)(k^{-1/\alpha}+r)$ $(j=1, \ldots, k)$ and $\xi'=k^{-1/\beta}=\xi$. Let A be the homogeneous perfect set of type $(n, \eta_1, \eta_2, \ldots, \eta_n, \xi)$ and A_q be the qth approximation of A. Similarly B be the homogeneous perfect set of type $(k, \eta_1', \eta_2', \ldots, \eta_k', \xi')$ and B_q be the qth approximation of B. Then Lemma 3 and Corollary 1 to Theorem 1 show that $0<\Lambda_\alpha(A)<\infty$, $0<\Lambda_\beta(B)<\infty$, $\underline{D}^{(\alpha)}(A, a)>0$ for any a in A and $\underline{D}^{(\beta)}(B, b)>0$ for any b in B. Hence Lemma 2 implies that $0<\Lambda_{\alpha+\beta}(A\times B)$. According to $n^{-1/\alpha}=k^{-1/\beta}$ the set $A\times B$ is covered by $(nk)^q$ closed cubes with side $n^{-q/\alpha}$. It follows that $\Lambda_{\alpha+\beta}(A\times B)\leqq \lim_{q\to\infty}(nk)^qn^{-q(\alpha+\beta)/\alpha}=1$. Therefore $\Lambda_{\alpha+\beta}(A\times B)<\infty$. Hence it is sufficient to prove that $\mathrm{proj}_{\pi/4}(A\times B)$ has positive $(\alpha+\beta)$-dimensional Hausdorff measure.

By the inequlity $(1-n^{-1/\alpha})(n^{-1/\alpha}+n^{-2/\alpha}+\ldots)+r>n^{-1/\alpha}$, there exists the smallest positive integer q_0 such that $(1-n^{-1/\alpha})(n^{-1/\alpha}+n^{-2/\alpha}+\ldots+n^{-(q_0-1)/\alpha})+r>n^{-1/\alpha}$. Since $r>n^{-q_0/\alpha}$, $\mathrm{proj}_{\pi/4}(A_{q_0}\times B_{q_0})$ consists of nk disjoint closed intervals with length $2^{-1/2}(n^{-1/\alpha}+n^{-q_0/\alpha})$ on the line $y=-x$. Similarly being shrinked with the same ratio $n^{-1/\alpha}$, $\mathrm{proj}_{\pi/4}(A_{q_0+1}\times B_{q_0+1})$ consists of $(nk)^2$ disjoint closed intervals with $2^{-1/2}n^{-1/\alpha}$

$(n^{-1/\alpha}+n^{-q_0/\alpha})$ on the line $y=-x$. It follows that $\mathrm{proj}_{\pi/4}(A\times B)=\bigcap_{q=1}^{\infty} C_q$, where C_q is the union of $(nk)^q$ disjoint closed intervals on the line $y=-x$ with length $2^{-1/2}n^{-(q-1)/\alpha}(n^{-1/\alpha}+n^{-q_0/\alpha})$. By a method similar to that in the proof of Theorem 1, we obtain $\Lambda_{\alpha+\beta}(\mathrm{proj}_{\pi/4}(A\times B))>0$. Thus we obtain $\Lambda_{\alpha+\beta}(A-B)>0$.

For given α, β such that $0<\alpha\leqq\beta<1$ and $\alpha+\beta\leqq 1$ we take an increasing sequence $\{\alpha_q\}_{q=1}^{\infty}$ which satisfies $0<\alpha_q<\alpha$, $\lim_{q\to\infty}\alpha_q=\alpha$, $\alpha_q=\beta\log n_q/\log k_q$ for some integers n_q and k_q, $2\leqq n_q\leqq k_q$. This is possible because the set $\{\log n/\log k\,;\, n,\ k \text{ integers } 2\leqq n\leqq k\}$ is dense in $[0,1]$. Then as constructed above, there exist sets $A_q\subset[0,1]$ and $B_q\subset[0,1]$ for each q such that $0<\Lambda_{\alpha_q}(A_q)<\infty$, $0<\Lambda_\beta(B_q)<\infty$, $0<\Lambda_{\alpha_q+\beta}(A_q-B_q)<\infty$ and $\underline{D}^{\beta)}(B_q,\ b)>0$ for every b in B_q. Set $A=\bigcup_{q=1}^{\infty}A_q$ and $B=\bigcup_{q=1}^{\infty}B_q$. Then $A-B=\bigcup_{q,q'}^{\infty}(A_q-B_{q'})$. By Lemma 2, $\dim A_q\times B_{q'}=\alpha_q+\beta<\alpha+\beta$ for any q, $q'\geqq 1$. Hence $\Lambda_{\alpha+\beta}(A_q-B_{q'})=0$ for any q, $q'\geqq 1$. Therefore

$$\Lambda_{\alpha+\beta}(A-B)\leqq\sum_{q,q'=1}^{\infty}\Lambda_{\alpha+\beta}(A_q-B_{q'})=0.$$

Moreover, $\Lambda_{\alpha_q+\beta}(A-B)\geqq\Lambda_{\alpha_q+\beta}(A_q-B_q)>0$ for each q, so that $\dim(A-B)=\alpha+\beta$. Thus these sets A and B are required ones.

For any α, β such that $0<\alpha\leqq\beta<1$ and $1<\alpha+\beta$, we take a positive number $\alpha_0=1-\beta$. For α_0 and β we can construct sets A_1 and B which $\dim A_1=\alpha_0$, $\dim B=\beta$, $\dim A_1\times B=\alpha_0+\beta=1$ and $\dim(A_1-B)=1$. Let A_2 be the one-dimensional generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty},\ \{\lambda_q\}_{q=1}^{\infty}]$, where $k_q=2$ and $\lambda_q=2^{-q/\alpha}$ $(q\geqq 1)$. Set $A=A_1\cup A_2$. Then these sets A and B have the required properties. Thus we have proved the theorem.

REMARK. In case $\beta=\alpha(\log k/\log n)$, $0<\alpha+\beta<1$, where k and n are integers $2\leqq n\leqq k$ and k is a multiple of n, we can construct sets which attain the lower bound of the theorem.

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