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Note on Self-Homotopy-Equivalences of the Twisted Principal Fibrations

Kouzou TSUKIYAMA*

Abstract : Let X be a connected CW-complex. The group $G_{\#}(X)$ of all based homotopy classes of self-homotopy-equivalences of $(X, *)$ inducing the identity automorphisms of all homotopy groups is studied, and we obtain the following exact sequence concerning the twisted principal fibration $p : \overline{P}_\theta \longrightarrow B$ with fibre $K(G, n)$

$$H^n(B ; G) \longrightarrow G_{\#}(\overline{P}_\theta) \longrightarrow G_{\#}(B),$$

where $H^n(B ; G)$ is the cohomology with the local coefficient induced by $q\theta : B \longrightarrow L = L_\phi(G, n+1) \longrightarrow K = K(\pi_1(B), 1)$.

Introduction

Let X be a connected CW-complex with base point $*$. Then, we have considered the group $G_{\#}(X)$ of all based homotopy classes of self-homotopy-equivalences of $(X, *)$ inducing the identity automorphisms of all homotopy groups (cf. [1], [11]).

The purpose of this note is to establish the exact sequences of $G_{\#}(X)$ concerning the twisted principal fibrations which generalize the exact sequences of M. Arkowitz-C. R. Curjel [1] and Y. Nomura [6].

In §1, we review the twisted principal fibrations (of. [5, §§2-3]), and in §2, we study the Postnikov-system by using the theorems of J. F. McClendon [4], and in §3, we prove the above exact sequences of $G_{\#}(X)$.

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§1. Twisted principal fibrations

Let Z be a given based space. A Z-space $A = (A, f)$ is a based space A together with a based map $f : A \longrightarrow Z$. For two Z-spaces $A = (A, f)$ and $B = (B, g)$, the pull back

$$A \times_Z B = \{a, b) \mid f(a) = g(b)\} \subset A \times B$$

of A and B is a Z-space with a map $(f, g) : A \times_Z B \longrightarrow Z$, $(f, g)(a, b) = f(a) = g(b)$. A based map $h : (A, *) \longrightarrow (B, *)$ is a Z-map if $gh = f$, and a homotopy $h_t : (A, *) \longrightarrow (B, *)$ is a Z-homotopy if $gh_t = f$ for all t, and $[A, B]_Z$ denotes the set of Z-homotopy classes of Z-maps of A to B.

Now, let G be an abelian group, π be a group, and $\phi : \pi \longrightarrow \mathrm{Aut}\, G$ be a given homomorphism. Then, there is an associated homomorphism $\phi : \pi \longrightarrow Homco\,(K(G, n+1), *)$, where $K(G, n+1)$ is an Eilenberg-MacLane CW-complex. And considering the Eilenberg-MacLane CW-complex $K = K(\pi, 1)$, the universal covering $\widetilde{K} \longrightarrow K$, and the usual action of π on $\widetilde{K}$, we have the fibre bundle

$$(1.1)\quad K(G, n+1) \longrightarrow L_\phi(G, n+1) = \widetilde{K} \times_\pi K(G, n+1) \xrightarrow{q} K = K(\pi, 1)$$

with structure group π. Since $\widetilde{K} \times_\pi * = K$, we have the canonical cross section $s : K \longrightarrow \widetilde{K} \times_\pi K(G, n+1)$ such that $s(K) = \widetilde{K} \times_\pi *$.

Let μ be the usual multiplication on $K(G, n+1)$. Then, for the K-space $L_\phi(G, n+1)$ of (1.1), we have the K-map

* Department of Mathematics, Faculty of Education, Shimane University, Matsue, Japan

(1. 2) $\mu_\phi : L_\phi(G, n+1)\times_K L_\phi(G, n+1) \longrightarrow L_\phi(G, n+1)$ by

$$\mu_\phi([\tilde{k}, x], [\tilde{k}, x']) = [\tilde{k}, \mu(x, x')]$$

and we have the following

LEMMA 1. 3. ([5, p. 7]) *Let X have the based homotopy type of a CW-complex* and *X be a K-space with a map $u : X \longrightarrow K$. Then, the K-homotopy set $[X, L_\phi(G, n+1)]_K$ is an abelian group by the multiplication $[f]+[g] = [\mu_\phi(f\times g)\Delta]$, where $\Delta : X \longrightarrow X\times X$ is the diagonal map and* $[X, L_\phi(G, n+1)]_K$ is isomorphic to *$H^{n+1}(X ; G)$, where $H^{n+1}(X ; G)$ is the cohomology with the local coefficient induced by $u : X \longrightarrow K$.*

Put $L = L_\phi(G, n+1)$. And consider the following path spaces

$\bar{P}L = \{\lambda : I \longrightarrow L \mid \lambda(0) \in s(K),\ q\lambda(0) = q\lambda(t) \text{ for all } t \in I\}$

$\bar{\Omega}L = \{\lambda \in \bar{P}L \mid \lambda(0) = \lambda(1)\}$

Then, we have the following

Lemma 1. 4. *The projection*

$r : \bar{P}L \longrightarrow L,\ r(\lambda) = \lambda(1)$

is a fibration with fibre $\Omega K(G, n+1) = K(G, n)$. Furthermore,

$qr : \bar{P}L \longrightarrow K$ and $qr : \bar{\Omega}L \longrightarrow K$

are fibrations with fibre $PK(G, n+1)$ and $\Omega K(G, n+1) = K(G, n)$, where $PK(G, n+1)$ and $\Omega K(G, n+1)$ are the ordinary path space and loop space of $K(G, n+1)$.

On the other hand, the given homomorphism $\phi : \pi \longrightarrow Homeo\,(K(G, n+1), *)$ induces the homomorphism

$\phi' : \pi \longrightarrow Homeo\,(\Omega K(G, n+1), *),\ \phi'(g)(\lambda)(t) = \phi(g)(\lambda(t))$.

And we have the fibration

$q' : L_{\phi'}(G, n) \longrightarrow K$

with fibre $\Omega K(G, n+1) = K(G, n)$ admitting the canonical cross section s' by (1. 1).

We have the natural K-homeomorphism

(1. 5) $\psi : L_{\phi'}(G, n) \longrightarrow \bar{\Omega}L_\phi(G, n+1),\ \psi([\tilde{k}, \lambda])(t) = [\tilde{k}, \lambda(t)]$,

which satisfies $qr\psi = q'$. And let

$\chi : \bar{\Omega}L \times_K \bar{\Omega}L \longrightarrow \bar{\Omega}L$

be given by the join of loops. Then, we have the following

LEMMA 1. 6. *The natural K-homeomorphism $\psi : L_{\phi'}(G, n) \longrightarrow \bar{\Omega}L_\phi(G, n+1)$ induces an isomorphism*

$\psi_* : [X, L_{\phi'}(G, n)]_K \longrightarrow [X, \bar{\Omega}L_\phi(G, n+1)]_K$

for any K-space X, where the domain is an abelian group of Lemma 1. 3, *and the multiplication in the range is induced by χ mensioned as above.*

Now, let B be a CW-complex, and $\theta : B \longrightarrow L_\phi(G, n+1)$ be a given based map, where the base point of $L_\phi(G, n+1)$ is taken to be $* \in s(K) \subset L_\phi(G, n+1)$. Then, from the fibration $r : \bar{P}L \longrightarrow L\ (L = L_\phi(G, n+1))$, θ induces a fibration

$p : \bar{P}_\theta = B \times_L \bar{P}L \longrightarrow B$

with fibre $\Omega K(G, n+1) = K(G, n)$, which is called the twisted principal fibration with classifying map θ.

$$\begin{array}{ccccc} \bar{P}_\theta & \longrightarrow & \bar{P}L & & \bar{\Omega}L \\ \Big\downarrow p & & \Big\downarrow r & & \Big\downarrow qr \\ B & \xrightarrow{\ \theta\ } & L & \xrightarrow{\ q\ } & K \end{array}$$

We say that the based topological space X is homotopy-well-pointed if the inclusion

$\{*\} \subset X$ is a homotopy-cofibration (cf. [3, p. 164 and §2]).

LEMMA 1.7. *$\bar{P}_0$ is homotopy-well-pointed.*

PROOF. Since $K(G, n+1)$ is a CW-complex, $\Omega K(G, n+1) = K(G, n)$ has the based homotopy type of a CW-complex, and every point of a CW-complex is homotopy-well-pointed (cf. [8, p. 380]), $\{*\} \subset K(G, n+1)$ is a homotopy-cofibration (cf. [3, p. 46, Korollar (2.7]), and by the theorem of A. Strøm ([10, p. 141, Theorem 12]), $p^{-1}(*) = K(G, n+1) \subset \bar{P}_0$ is a cofibration, since $\{*\}$ is closed in B. By the fact that the composition of two homotopy-cofibrations is a homotopy-cofibration (cf. [3, p. 44, Satz (2.4)], $\{*\} \subset \bar{P}_0$ is a homotopy-cofibration, that is, $\bar{P}_0$ is homotopy-well-pointed. *q. e. d.*

We define the K-map $\nu : \bar{\Omega} L \times_K \bar{P}_0 \longrightarrow \bar{P}_0$ by the relation $\nu(m, (b, n)) = (b, m \vee n)$, where $m \vee n$ is the ordinary path addition in L, where the base point of $\bar{\Omega} L$ is the constant loop at $* \in L$. Then, ν defines the following action for any K-space X.

(1.8) $\nu_* : [X, \bar{\Omega} L]_K \times [X, \bar{P}_0]_K \longrightarrow [X, \bar{P}_0]_K$.

Let $p_* : [X, \bar{P}_0]_K \longrightarrow [X, B]_K$, then we have the following

PROPOSITION 1.9. (cf. [5, p. 6, Lemma]) *$p_*(\alpha) = p_*(\beta)$ $(\alpha, \beta \in [X, \bar{P}_0]_K)$ if and only if there exists $\delta \in [X, \bar{\Omega} L]_K$ such that $\nu_*(\delta, \beta) = \alpha$.*

§2. Postnikov-system

Let X be a connected CW-complex, and $\{X_n\}$ be the Postnikov-system of X (cf. [11, pp. 218-219]). And let $\phi : \pi_1(X) \longrightarrow \mathrm{Aut}\, \pi_n(X)$ be the local coefficient system associated with $p_n : X_n \longrightarrow X_{n-1}$ (this becomes an usual action of $\pi_1(X)$ on $\pi_n(X)$), and let the associated homomorphism $\phi : \pi_1(X) \longrightarrow \mathit{Homeo}\, (K(\pi_n(X), n+1), *)$. Put $L = L_\phi(\pi_n(X), n+1)$ and $K = K(\pi_1(X), 1)$, then we have the following

PROPOSITION 2.1. *For the fibration $p_n : X_n \longrightarrow X_{n-1}$ there exist maps $k : (X_{n-1}, *) \longrightarrow (L, *)$, and $\eta : (X_n, *) \longrightarrow (\bar{P}_k, *)$ such that η is a based homotopy-equivalence.*

Therefore, $\mathcal{G}_\#(X_n) = \mathcal{G}_\#(\bar{P}_k)$.

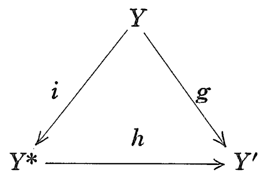
PROOF. Let $i_n \in H^n(K(\pi_n(X), n), \pi_n(X))$ be the fundamental cohomology class of $K(\pi_n(X), n)$. Then, by J. F. McClendon ([4, Theorem 4.1 and §§2-3]), there exist maps $k : (X_{n-1}, *) \longrightarrow (L, *)$ such that $[k] \in [X_{n-1}, L]_K = H^{n+1}(X_{n-1}, \pi_n(X))$ is the transgression image of i_n, where X_{n-1} is considered as a K-space by $p_2 \circ \cdots \circ p_{n-2} p_{n-1} : X_{n-1} \longrightarrow X_1 = K(\pi_1(X), 1) = K$, and $\eta : (X_n, *) \longrightarrow (\bar{P}_k, *)$ such that $p\eta = p_n$ and $\eta \mid K(\pi_n(X), n) \simeq i_n$ rel $*$, where p is the twisted principal fibration induced by k.

$$\begin{array}{ccccc}
K(\pi_n(X), n) & \longrightarrow & X_n & & \\
\downarrow i_n & & \downarrow \eta & & \\
K(\pi_n(X), n) & \longrightarrow & \bar{P}_k & \longrightarrow & \bar{P}L \\
& & \downarrow p & & \downarrow \\
& & X_{n-1} & \xrightarrow{\;k\;} & L
\end{array}$$

Now, since $i_{n*} : \pi_n(K(\pi_n(X), n)) \longrightarrow \pi_n(K(\pi_n(X), n))$ is an isomorphism, $\eta_* : \pi_i(X_n) \longrightarrow \pi_i(\bar{P}_k)$ is an isomorphism for every $i \geqq 1$. Since $\bar{P}_k$ has the free homotopy type of a CW-complex (cf. [9, Prop. (0) and [7, Theorem 2]), η is a free homotopy-equivalence by the theorem of J. H. C. Whitehead. Since $\overline{P}_k$ is homotopy-well-pointed by lemma 1. 7 and X_n is homotopy-well-pointed (cf. [3, p. 46, Korollar (2. 7) and [8, p. 380]), η is a based homotopy-equivalence (cf. [3, p. 54, Satz (2. 18)]).

Therefore, $G_{\#}(X_n) = G_{\#}(\overline{P}_k)$. *q. e. d.*

PROPOSITION 2. 2. *Let Y be a topological space which has the free homotopy type of a connected CW-complex, whose base point is homotopy-well-pointed, and let Y^* be a topological space obtained by attaching $(i+1)$-cells $(i \geqq n)$ to Y, so that Y^* kills the homotopy groups $\pi_i(Y)$ for every $i \geqq n$. Then, for any CW-complex Y' with $\pi_i(Y') = 0$ for every $i \geqq n$, and for any map $g : (Y, *) \longrightarrow (Y', *)$, there exists a map $h : (Y^*, *) \longrightarrow (Y', *)$ such that $hi \simeq g$ rel $*$ (i is the inclusion of Y to Y^*). Such two maps are homotopic rel $*$, and h does not depend on the based homotopy class of g. Furthermore, if $g_* : \pi_i(Y) \longrightarrow \pi_i(Y')$ is an isomorphism for every $i < n$, then, h is a based homotopy-equivalence.*



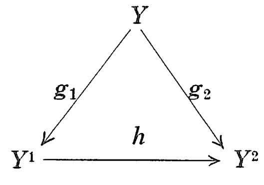
PROOF. Since (Y^*, Y) is a relative CW-complex with dimension $(Y^*, Y) \geqq n+1$, and $\pi_i(Y') = 0$ for every $i \geqq n$, there is an extension h_0 of g to Y^*, and such extensions are homotopic rel Y by the elementary homotopy theory. Also, for any map $h : (Y^*, *) \longrightarrow (Y', *)$ such that $hi \simeq g$ rel $*$, there exists $h' : (Y^*, *) \longrightarrow (Y', *)$ such that $h \simeq h'$ rel $*$, and $h' i = g$ by the homotopy extension theorem.

Therefore, $h' \simeq h_0$ rel Y, and consequently, $h \simeq h_0$ rel $*$. It is trivial that h does not depend on the based homotopy class of g. Furthermore, if $g_* : \pi_i(Y) \longrightarrow \pi_i(Y')$ is an isomorphism for every $i < n$, then, $h_* : \pi_i(Y^*) \longrightarrow \pi_i(Y')$ is an isomorphism for every $i \geqq 1$, since $\pi_i(Y^*) = \pi_i(Y') = 0$ for every $i \geqq n$.

By assumption, Y^* has the free homotopy type of a connected CW-complex. Therefore, h is a free homotopy-equivaleuce by the theorem of J. H. C. Whitehead. Since Y is homotopy-well-pointed and $Y \subset Y^*$ is a cofibration, Y^* is homotopy-well-pointed (cf. [3, p. 44, Satz (2. 4)]). Y' is homotopy-well-pointed (cf. [8, p. 380])

Therefore, h is a based homotopy-equivalence (cf. [3, p. 54, Satz (2. 18)]). *q. e. d.*

PROPOSITION 2. 3. *Let Y be as in Proposition 2. 2, Y^1 and Y^2 be connected CW-complexes such that $\pi_i(Y^1) = \pi_i(Y^2) = 0$ for every $i \geqq n$, and g_j be a map of $(Y, *)$ to $(Y^j, *)$ such that $g_{j*} : \pi_i(Y) \longrightarrow \pi_i(Y^j)$ is an isomorphism for every $i < n$ $(j = 1, 2)$. Then, there exists a based homotopy-equivalence $h : (Y^1, *) \longrightarrow (Y^2, *)$ such that $hg_1 \simeq g_2$ rel $*$, and such two homotopy-equivalences are homotopic rel $*$.*



PROOF. By Proposition 2. 2, there exists a based homotopy-equivalence $h_j : (Y^*, *) \longrightarrow (Y^j, *)$ such that $h_j i \simeq g_j$ rel $*$ $(j = 1, 2)$, since $g_{j*} : \pi_i(Y) \longrightarrow \pi_i(Y^j)$ is an isomorphism for every $i < n$ $(j = 1, 2)$. Now, let h_1' be the based homotopy-inverse of h_1. Then, $h_2 h_1' : (Y^1, *) \longrightarrow (Y^2, *)$ satisfies $(h_2 h_1') \simeq g_2$ rel $*$. Also, if $h : (Y^1, *) \longrightarrow (Y^2, *)$ satisfies $hg_1 \simeq g_2$ rel $*$, then, $hh_1 : (Y^*, *) \longrightarrow (Y^2, *)$ satisfies $(hh_1) i \simeq g_2$ rel $*$. Hence, by Proposition 2. 2, $hh_1 \simeq h_2$ rel $*$, that is, $h \simeq h_2 h_1'$ rel $*$. Now, $h_2 h_1'$ is a based homotopy-equivalence, so is tha map h. *q. e. d.*

Now, assuming $\pi_i(B) = 0$ for every $i \geqq n$, then, $p_* : \pi_i(\overline{P}_\theta) \longrightarrow \pi_i(\mathrm{B})$ is an isomorphism for every $i < n$, $\pi_n(\overline{P}_\theta) = \pi_n(K(G, n)) = G$, and $\pi_i(\overline{P}_\theta) = 0$ for every $i \geqq n+1$.

$$\begin{array}{ccc} \overline{P}_\theta & \xrightarrow{\ f\ } & \overline{P}_\theta \\ \downarrow p & & \downarrow p \\ B & \xrightarrow{\ f'\ } & B \end{array}$$

By Propositions 2. 2 and 2. 3, given a based self-homotopy-equivalence $f : (\overline{P}_\theta, *) \longrightarrow (\overline{P}_\theta, *)$ there exists a unique based self-homotopy-equivalence up to based homotopy $f' : (B, *) \longrightarrow (B, *)$ such that $pf \simeq f'p$ rel $*$, and f' does not depeed on the based homotopy class of f. Therefore, we can define the natural homomorphism $J : G_\#(\overline{P}_\theta) \longrightarrow G_\#(B)$.

§3. Proof of the theorems

Let $\phi : \pi_1(B) \longrightarrow Aut\ G$ be the local coefficient system on B. Put $L = L_\phi(G, n+1)$ and $K = K(\pi_1(B), 1)$. By J. F. McClendon ([5, pp. 3-4, Theorem]), Lemma 1. 3 and Lemma 1. 6, we have the following exact sequence of the local coefficient cohomology.

$$\begin{array}{ccccccc} H^n(F ; G) & \xleftarrow{\ i^*\ } & H^n(\overline{P}_\theta ; G) & \xleftarrow{\ p^*\ } & H^n(B ; G) & \longleftarrow 0 & (F = K(G, n)) \\ \| & & \| & & \| & & \\ [F, F] & \xleftarrow{\ i^*\ } & [\overline{P}_\theta, \Omega L]_K & \xleftarrow{\ p^*\ } & [B, \overline{\Omega}L]_K & \longleftarrow 0 & \end{array} \tag{3. 1}$$

We define a map $\Delta : Ker\ [i^* : [\overline{P}_\theta, \overline{\Omega}L]_K \longrightarrow [F, F]] \longrightarrow G_\#(\overline{P}_\theta)$. By (1. 8), take $X = \overline{P}_\theta$ and let $\Delta(a) = \nu_*\{a, 1_{\overline{P}_\theta}\}$, that is, $\Delta(a) : \overline{P}_\theta \xrightarrow{\{a, 1\}} \overline{\Omega}L \times_K \overline{P}_\theta \xrightarrow{\ \nu\ } \overline{P}_\theta$.

Note that the restriction $\overline{\nu} = \nu\,|_{F \times F}$ is the ordinary multiplication on $\Omega K(G, n+1) = K(G, n) = F$. We have the following commutative diagram.

$$\begin{array}{ccccc} K(G, n) & \xrightarrow{\{a, 1\}} & K(G, n) \times K(G, n) & \xrightarrow{\ \overline{\nu}\ } & K(G, n) \\ i \downarrow & & i \downarrow & & i \downarrow \\ \overline{P}_\theta & \xrightarrow{\{a, 1\}} & \overline{\Omega}L \times_K \overline{P}_\theta & \xrightarrow{\ \nu\ } & \overline{P}_\theta \\ & & \text{projection} \downarrow & & \\ p \downarrow & & \overline{P}_\theta & & p \downarrow \\ & & p \downarrow & & \\ B & \xrightarrow{\ id\ } & B & \xrightarrow{\ id\ } & B \end{array} \tag{3. 3}$$

By the above commutative diagram, $(\nu_*\{a, 1\})_* : \pi_i(\overline{P}_\theta, *) \longrightarrow \pi_i(\overline{P}_\theta, *)$ is the identity automorphism for every $i \geqq 1$ if $a \in Ker\ [i^* : [\overline{P}_\theta, \overline{\Omega}L]_K \longrightarrow [F, F]]$. Since $\overline{P}_\theta$ is homotopy-well-pointed by Lemma 1. 7, $\nu_*\{a, 1\}$ is a based homotopy-equivalence (cf. [3, p. 54, Satz (2. 18)]), so the map Δ is defined. Thus we have the following sequence of groups and maps.

$$H^n(B ; G) \xrightarrow{\ \ \Delta\ \ } G_\#(\overline{P}_\theta) \xrightarrow{\ \ J\ \ } G_\#(B), \tag{3. 3}$$

where the action of $\pi_1(B)$ on G is induced by $q\theta : B \longrightarrow L \longrightarrow K$.

THEOREM 3. 4. *Im* Δ = *Ker* J.

PROOF. *Im* $\Delta \subset$ *Ker* J is evident by the diagram (3. 2), and we will show that *Im* $\Delta \supset$ *Ker* J. We assume that $J(g) = 1$, that is, $pg \simeq p1_{\overline{P}_\theta}$ rel $*$. By Proposition 1. 9, there exists $\delta \in [\overline{P}_\theta, \overline{\Omega}L]_K$ such that $\nu_*\{\delta, 1_{\overline{P}_\theta}\} = g$, that is, $\Delta(\delta) = g$.

Then, $\nu\{\delta i, i\} = \nu\{\delta, 1\}i \simeq gi \simeq i1_F = i$ rel $*$, and on $\pi_n(F) = G$, $\nu_*(a, b) = a+b$ $(a, b \in G)$, so $\nu_*\{\delta i, i\}_*(a) = (\delta i)_*(a) + i_*(a) = i_*(a)$. Hence $(\delta i)_* = 0$, that is, $\delta \in Ker\ i^* = Im\ p^* = H^n(B ; G)$.

q. e. d.

THEOREM 3. 5. *Δ is a homomorphism of groups.*

PROOF. Let $\mu_{\phi'}: L_{\phi'} \times_K L_{\phi'} \longrightarrow L_{\phi'}$ be the K-map as in (1. 2), and define $\Delta(\omega_1) \perp \omega_2 = \omega_2 \Delta(\omega_1)^{-1} \in [\bar{P}_\theta, \bar{\Omega}L]_K$ ($\omega_1, \omega_2 \in Ker\ i^* \subset [\bar{P}_\theta, \bar{\Omega}L]_K$). Since

$$\begin{aligned}\mu_{\phi'}((\Delta(\omega_1) \perp \omega_2)\Delta(\omega_1), \omega_1) &= \mu_{\phi'}(\omega_2\Delta(\omega_1)^{-1}\Delta(\omega_1), \omega_1)\\ &= \mu_{\phi'}(\omega_2, \omega_1)\\ &= \omega_2 + \omega_1,\end{aligned}$$

We have

$$\begin{aligned}\Delta(\omega_2+\omega_1) &= \nu_*\{\omega_2+\omega_1, 1_{\bar{P}_\theta}\}\\ &= \nu_*\{\mu_{\phi'}((\Delta(\omega_1) \perp \omega_2)\Delta(\omega_1), \omega_1), 1_{\bar{P}_\theta}\}\\ &= \nu_*\{(\Delta(\omega_1) \perp \omega_2)\Delta(\omega_1), \nu_*\{\omega_1, 1_{\bar{P}_\theta}\}\} \quad \text{(by Lemma 1. 6)}\\ &= \nu_*\{(\Delta(\omega_1) \perp \omega_2)\Delta(\omega_1), \Delta(\omega_1)\}\\ &= \nu_*\{\Delta(\omega_1) \perp \omega_2, 1_{\bar{P}_\theta}\}\Delta(\omega_1)\\ &= \Delta(\Delta(\omega_1) \perp \omega_2)\Delta(\omega_1).\end{aligned}$$

Now, $\omega_2 = \omega_2' p$ for some $\omega_2' \in [B, \bar{\Omega}L]_K$ by (3. 1), we have

$$\begin{aligned}\Delta(\omega_1) \perp \omega_2 &= \omega_2' p\Delta_1(\omega_1)^{-1}\\ &= \omega_2' J(\Delta(\omega_1)^{-1})p\\ &= \omega_2' p\\ &= \omega_2.\end{aligned}$$

Therefore, $\Delta(\omega_2+\omega_1) = \Delta(\omega_2)\Delta(\omega_1)$. *q. e. d.*

By (1. 8) take $X = \bar{P}_\theta$ and let $I(1_{\bar{P}_\theta})$ be the isotropy group of $1_{\bar{P}_\theta}: \bar{P}_\theta \longrightarrow \bar{P}_\theta$ under the action of $[\bar{P}_\theta, \bar{\Omega}L]_K$ on $[\bar{P}_\theta, \bar{P}_\theta]_K$.

Trivially $I(1_{\bar{P}_\theta})$ is contained in $Ker\ [i^*: [\bar{P}_\theta, \bar{\Omega}L] \longrightarrow [F, F]]$, and we have the following theorem.

THEOREM 3. 6. *Assume that a connected CW-complex B satisfies $\pi_i(B) = 0$ for every $i \geq n$, let $\phi: \pi_1(B) \longrightarrow Aut\ G$ be the local coefficient system on B, and let $p: \bar{P}_\theta \longrightarrow B$ be the twisted principal fibration with fibre $K(G, n)$. Then, the following sequence of groups and homomorphism is exact.*

$$1 \longrightarrow I(1_{\bar{P}_\theta}) \xrightarrow{\subset} H^n(B; G) \xrightarrow{\Delta} \mathcal{G}_\#(\bar{P}_\theta) \xrightarrow{J} \mathcal{G}_\#(B),$$

where $H^n(B; G)$ is the cohomology with the local coefficient induced by $q\theta: B \longrightarrow L = L_\phi(G, n+1) \longrightarrow K = K(\pi_1(B), 1)$.

By Propositions 2. 1, 2. 2 and 2. 3, the following is obtained.

COROLLARY 3. 7. *Let X be a connected CW-complex and $\{X_n\}$ be a Postnikov-system of X. Then, the following sequence of groups and homomorphisms is exact for every $n \geq 1$.*

$$1 \longrightarrow I(1_{X_n}) \xrightarrow{\subset} H^n(X_{n-1}; \pi_n(X)) \xrightarrow{\Delta} \mathcal{G}_\#(X_n) \xrightarrow{J} \mathcal{G}_\#(X_{n-1}),$$

where $H^n(X_{n-1}; \pi_n(X))$ is the cohomology with the local coefficient $\{\pi_n(X)\}$, where $\pi_1(X_{n-1}) = \pi_1(X)$ acts on $\pi_n(X)$ as usual, and $I(1_{X_n})$ is the isotropy group of $1_{X_n}: X_n \longrightarrow X_n$ under the action of $[X_n, \bar{\Omega}L]_K$ on $[X_n, X_n]_K$ which is defined in the following diagram.

$$\begin{array}{ccccc}
[\bar{P}_k, \bar{\Omega}L]_K & \times & [\bar{P}_k, \bar{P}_k]_K & \xrightarrow{\nu_*} & [\bar{P}_k, \bar{P}_k]_K\\
\uparrow{\scriptstyle \eta^*} & & \uparrow{\scriptstyle \eta^*} & & \uparrow{\scriptstyle \eta^*}\\
[X_n, \bar{\Omega}L]_K & \times & [X_n, \bar{P}_k]_K & \xrightarrow{\nu_*} & [X_n, \bar{P}_k]_K\\
\| & & \uparrow{\scriptstyle \xi^*} & & \uparrow{\scriptstyle \xi^*}\\
[X_n, \bar{\Omega}L]_K & \times & [X_n, X_n]_K & \longrightarrow & [X_n, X_n]_K,
\end{array}$$

where ξ *is the based homotopy-inverse of* $\eta : X_n \longrightarrow \bar{P}_k$ *by Proposition* 2. 1.

Furthermore, $Im\ J = \{[f_{n-1}] \in G_{\#}(X_{n-1}) \mid [k] = f_{n-1}{}^{*}[k]\}$, *where* $[k] \in H^{n+1}(X_{n-1}\,;\,\pi_n(X))$ *by Proposition* 2. 1.}

PROOF. (cf. [4, p. 4]) By Proposition 2. 1 $[k] = \delta(i_n)$, and the following diagram commutes.

$$\begin{array}{ccc} H^n(K(\pi_n(X), n)\,;\,\pi_n(X)) & \xrightarrow{\ \delta\ } & H^{n+1}(X_{n-1}\,;\,\pi_n(X)) \\ & \searrow^{\delta} & \downarrow f_{n-1}{}^{*} \\ & & H^{n+1}(X_{n-1}\,;\,\pi_n(X)) \end{array}$$

Hence, if $[f_{n-1}]$ is an image under J, it belongs to the right hand side. On the other hand, Suppose that $[f_{n-1}]$ belongs to the right hand side. Then, the following diagram is based homotopy commutative,

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{\ f_{n-1}\ } & X_{n-1} \\ \downarrow k & & \downarrow k \\ L & \xrightarrow{\ id\ } & L \end{array}$$

and we may construct a based self-homotopy-equivalence $f_n : X_n \longrightarrow X_n$ such that $p_n f_n \simeq f_{n-1} p_n$ rel $*$ which induces the identity automorphism of $\pi_i(X_n)$ for any $i \geq 1$.

q. e. d.

COROLLARY 3. 8. (cf. [11, Theorem 1. 3]) *Assume that the connected CW-complex* X *satisfies* $\pi_i(X) = 0$ $(i > N)$ or *dim* $X = N$, *for some integer* N, *and that the cohomology groups of local coefficient are* $H^n(X_{n-1}\,;\,\pi_n(X)) = 0$ $(1 < n \leq N)$. Then, $G_{\#}(X) = 1$.

PROOF. Note that $G_{\#}(X_1) = G_{\#}(K(\pi_1(X), 1) = 1$. Then, we have $G_{\#}(X_n) = 1$ for every $n > 1$ by induction using Corollary 3. 7. *q. e. d.*

COROLLARY 3. 9. *Let* $\{X_n \mid n > 1\}$ *be as in Corollary* 3. 7. *If* $H^n(X_{n-1} : \pi_n(X))$ $(1 < n \leq N)$ *are finite (finitely generated) groups.* Then, $G_{\#}(X_N)$ *is a finite (finitely generated) group.*

COROLLARY 3. 10. *Let* X *be as in Corollary* 3. 8. *If Aut* $\pi_n(X)$ $(1 \leq n \leq N)$, *and* $H^n(X_{n-1}\,;\,\pi_n(X))$ $(1 < n \leq N)$ *are finite (finitely generated) groups. Then,* $G(X)$ (=*the group of all based homotopy classes of self-homotopy-equivalences of* $(X, *)$) *is a finite (finitely generated) group.*

PROOF. Consider the following exact sequence (cf. [1, p. 30, (*)]).

$$1 \longrightarrow G_{\#}(X) \longrightarrow G(X) \longrightarrow \sum_{1 \leq n \leq N} Aut\ \pi_n(X)$$

and use Corollary 3. 9. *q. e. d.*

Especially we obtain the following theorem.

Theorem 3. 11. (cf. [4, Theorem 3. 1]) *Let* X *be as in Corollary* 3. 8. *If* $\pi_n(X)$ $(1 \leq n \leq N)$ *are finite groups.*

Then, $G(X)$ is a finite group.

Proof. By ([2. p. 44, 15. 6]) we see that $H^2(\pi_1(X)\,;\,\pi_2(X))$ with the local coefficient is finite. And by induction using the local coefficient Serre spectral sequence (cf. [5, §1]), $H^n(X_{n-1}\,;\,\pi_n(X))$ $(1<n\leqq N)$ are finite groups. Hence by the above corollary $G(X)$ is a finite group. *q. e. d.*

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