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# A TRACE IDENTITY FOR PARABOLIC ELEMENTS OF $SL(2, \mathbf{C})$*

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Dedicated to the memory of Professor Nobuyuki Suita

## Introduction

The objective of this paper is to establish a trace identity for four parabolic elements in $SL(2, \mathbf{C})$. Let $F_{g,m}$ denote the orientable closed surface of genus $g$ with $m$ points removed. We assume that $m > 0$. The fundamental group $G_{g,m}$ of $F_{g,m}$ is generated by $2g + m$ elements

$$a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_m$$

satisfying a single relation

$$\left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) c_1 \cdots c_m = 1.$$

In [3] R. C. Penner introduced $\lambda$ length coordinates to the decorated Teichmüller spaces of punctured surfaces and proved the "ideal Ptolemy theorem". This theorem played an essential role in obtaining a faithful representation of the mapping class group $\mathcal{MC}_{g,m}$ as a group of rational transformations. Let $R_{g,m}$ denote the space of all conjugacy classes of faithful representations $\rho$ of $G_{g,m}$ in $SL(2, \mathbf{C})$ such that $\rho(c_i)$ is parabolic and $\operatorname{tr} \rho(c_i) = -2$ for $i = 1, 2, \dots, m$. In [2] the $\lambda$ length is complexified so as to parametrize the space $R_{g,m}$ and the "ideal Ptolemy theorem" is proved for a special case. The complexified $\lambda$ length is a kind of trace function and hence the "ideal Ptolemy theorem" proved in [2] is a trace identity (see Lemma 1.2 below). In this paper we establish another version of the "ideal Ptolemy theorem". Our main theorem is

Theorem 0.1. *Let $P_1$, $P_2$, $P_3$, $P_4$ be matrices in $SL(2, \mathbf{C})$ satisfying the following conditions*:

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1. $\mathrm{tr}\, P_1 = \mathrm{tr}\, P_2 = \mathrm{tr}\, P_3 = \mathrm{tr}\, P_4 = -2$, *and*
2. *Any two of them generate an irreducile group.*

*Let* $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \in SL(2, \mathbf{C})$ *be matrices with the following properties*:

$$P_1P_2 = -Q_1^2, \quad P_2P_3 = -Q_2^2, \quad P_3P_4 = -Q_3^2,$$
$$P_4P_1 = -Q_4^2, \quad P_3P_1 = -Q_5^2, \quad P_4P_2 = -Q_6^2$$

(*The existence and the uniqueness up to a multiple factor in* $\{-1, 1\}$ *of these matrices are shown in Lemma* 2.4). *If*

$$\mathrm{tr}\, Q_1Q_2Q_5 = \mathrm{tr}\, Q_2Q_3Q_6 = \mathrm{tr}\, Q_1Q_2Q_3Q_4 = -2,$$

*then the following equation holds*:

$$\mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_3 + \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_4 = \mathrm{tr}\, Q_5\, \mathrm{tr}\, Q_6. \tag{0.1}$$

Similar results hold for the cases where some pairs of $P_1$, $P_2$, $P_3$, $P_4$ generate reducible groups, see Section 4. In Section 5 we show an application of the trace identity (0.1) to the mapping class group $\mathcal{MC}_{0,4}$ acting on $R_{0,4}$.

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## 1. Trace functions

**1.1.** We shall use the following basic properties of traces of matrices without mentioning them:

$\mathrm{tr}\, Y^{-1}XY = \mathrm{tr}\, X$.

$\mathrm{tr}\, X_1X_2 \cdots X_n = \mathrm{tr}\, X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(n)}$ for any cyclic permutation $\sigma$ on $\{1, 2, \ldots, n\}$.

The group $SL(2, \mathbf{C})$ consists of matrices of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbf{C},\ ad - bc = 1$$

Each matrix of $SL(2, \mathbf{C})$ acts on the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ by a linear fractional transformation. We denote by $I$ the unit matrix in $SL(2, \mathbf{C})$. A matrix $A$ of $SL(2, \mathbf{C})$ other than $\pm I$ is called *parabolic* if $\mathrm{tr}\, A \in \{-2, 2\}$; *elliptic* if $-2 < \mathrm{tr}\, A < 2$. If, in particular, $\mathrm{tr}\, A = 0$, then $A^2 = -I$ and $A$ is called *elliptic of order* 2. We list some trace identities for $SL(2, \mathbf{C})$.

Lemma 1.1 ([1, 3.4]). *Matrices in* $SL(2, \mathbf{C})$ *satisfy the following equations.*

$$\mathrm{tr}\, A = \mathrm{tr}\, A^{-1}$$

$$\mathrm{tr}\, A\, \mathrm{tr}\, B = \mathrm{tr}\, AB + \mathrm{tr}\, AB^{-1} \tag{1.1}$$

$$\mathrm{tr}\, ABC = \mathrm{tr}\, A\, \mathrm{tr}\, BC + \mathrm{tr}\, B\, \mathrm{tr}\, CA + \mathrm{tr}\, C\, \mathrm{tr}\, AB - \mathrm{tr}\, A\, \mathrm{tr}\, B\, \mathrm{tr}\, C - \mathrm{tr}\, BAC \tag{1.2}$$

$$\begin{aligned}(1.3)\qquad 2 \operatorname{tr} ABCD = {} & \operatorname{tr} A \operatorname{tr} BCD + \operatorname{tr} B \operatorname{tr} ACD + \operatorname{tr} C \operatorname{tr} ABD + \operatorname{tr} D \operatorname{tr} ABC \\ & + \operatorname{tr} AB \operatorname{tr} CD - \operatorname{tr} AC \operatorname{tr} BD + \operatorname{tr} BC \operatorname{tr} DA \\ & - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} CD - \operatorname{tr} B \operatorname{tr} C \operatorname{tr} DA - \operatorname{tr} C \operatorname{tr} D \operatorname{tr} AB \\ & - \operatorname{tr} D \operatorname{tr} A \operatorname{tr} BC + \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C \operatorname{tr} D\end{aligned}$$

The following equation is obtained by a repeated use of (1.1).

$$(1.4)\qquad \operatorname{tr} A_1 \operatorname{tr} A_2 \cdots \operatorname{tr} A_n \operatorname{tr} A_1 A_2 \cdots A_n = \sum_{\varepsilon_i \in \{0,1\}} \operatorname{tr} A_1^{2\varepsilon_1} A_2^{2\varepsilon_2} \cdots A_n^{2\varepsilon_n}$$

LEMMA 1.2 ([2, Proposition 1.1]). *Let* $A, B, C, D \in SL(2, \mathbf{C})$ *and* $x = \operatorname{tr} A + \operatorname{tr} BCD$, $y = \operatorname{tr} B + \operatorname{tr} CDA$, $z = \operatorname{tr} C + \operatorname{tr} DAB$, $w = \operatorname{tr} D + \operatorname{tr} ABC$, $u = \operatorname{tr} AB + \operatorname{tr} CD$, $v = \operatorname{tr} BC + \operatorname{tr} AD$. *If* $\operatorname{tr} ABCD = -2$, *then*

$$xz + yw = uv.$$

LEMMA 1.3. *Let* $X, Y_1, \dots, Y_{n+1} \in SL(2, \mathbf{C})$, *where* $n \geq 1$. *If* $\operatorname{tr} Y_1 = \cdots = \operatorname{tr} Y_{n+1}$, *then*

$$\begin{aligned}&\sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_n} \operatorname{tr} XY_1^{\varepsilon_1} Y_2^{\varepsilon_1+\varepsilon_2} \cdots Y_n^{\varepsilon_{n-1}+\varepsilon_n} Y_{n+1}^{\varepsilon_n+1} \\ &\quad = \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_n} \operatorname{tr} XY_1^{\varepsilon_1+1} Y_2^{\varepsilon_1+\varepsilon_2} \cdots Y_n^{\varepsilon_{n-1}+\varepsilon_n} Y_{n+1}^{\varepsilon_n}.\end{aligned}$$

*Proof.* Let $a = \operatorname{tr} Y_1 = \cdots = \operatorname{tr} Y_n$. If $n = 1$, then

$$\operatorname{tr} XY_1 - \operatorname{tr} XY_1^2 Y_2 = \operatorname{tr} XY_1 + \operatorname{tr} XY_2 - a \operatorname{tr} XY_1 Y_2 = \operatorname{tr} XY_2 - \operatorname{tr} XY_1 Y_2^2.$$

For $n > 1$ we have

$$\begin{aligned}&\sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_n} \operatorname{tr} XY_1^{\varepsilon_1} Y_2^{\varepsilon_1+\varepsilon_2} \cdots Y_n^{\varepsilon_{n-1}+\varepsilon_n} Y_{n+1}^{\varepsilon_n+1} \\ &\quad = \sum_{\varepsilon_2,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_2+\cdots+\varepsilon_n} (\operatorname{tr} XY_2^{\varepsilon_2} Z Y_{n+1}^{\varepsilon_n+1} - \operatorname{tr}(Y_{n+1} X Y_1) Y_2^{1+\varepsilon_2} Z Y_{n+1}^{\varepsilon_n})\end{aligned}$$

where $Z$ stands the matrix $Y_3^{\varepsilon_2+\varepsilon_3} \cdots Y_n^{\varepsilon_{n-1}+\varepsilon_n}$. By induction and (1.1), the last term equals

$$\begin{aligned}&\sum_{\varepsilon_2,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_2+\cdots+\varepsilon_n} (\operatorname{tr} XY_2^{\varepsilon_2+1} Z Y_{n+1}^{\varepsilon_n} - \operatorname{tr}(Y_{n+1} X Y_1) Y_2^{\varepsilon_2} Z Y_{n+1}^{\varepsilon_n+1}) \\ &\quad = \sum_{\varepsilon_2,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_2+\cdots+\varepsilon_n} (\operatorname{tr} XY_2^{1+\varepsilon_2} Z Y_{n+1}^{\varepsilon_n} - a \operatorname{tr}(XY_1) Y_2^{\varepsilon_2} Z Y_{n+1}^{\varepsilon_n+1} \\ &\qquad + \operatorname{tr} XY_1 Y_2^{\varepsilon_2} Z Y_{n+1}^{\varepsilon_n})\end{aligned}$$

$$= \sum_{\varepsilon_2,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_2+\cdots+\varepsilon_n} (\mathrm{tr}\, XY_2^{1+\varepsilon_2} Z Y_{n+1}^{\varepsilon_n} - a\, \mathrm{tr}(XY_1)\, Y_2^{\varepsilon_2+1} Z Y_{n+1}^{\varepsilon_n} + \mathrm{tr}\, XY_1 Y_2^{\varepsilon_2} Z Y_{n+1}^{\varepsilon_n})$$

$$= \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_n}\, \mathrm{tr}\, XY_1^{\varepsilon_1+1} Y_2^{\varepsilon_1+\varepsilon_2} \cdots Y_n^{\varepsilon_{n-1}+\varepsilon_n} Y_{n+1}^{\varepsilon_n}. \qquad \square$$

LEMMA 1.4. *Let* $\{P_i, Q_i\}_{i=1}^n \subset SL(2,\mathbf{C})$, *where* $n \geq 2$, *be such that*

$$P_1P_2 = -Q_1^2, P_2P_3 = -Q_2^2, \dots, P_nP_1 = -Q_n^2$$

*and* $\mathrm{tr}\, P_1 = \mathrm{tr}\, P_2 = \cdots = \mathrm{tr}\, P_n$. *Then*

$$\mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3 \cdots \mathrm{tr}\, Q_n\, \mathrm{tr}\, Q_2Q_3\cdots Q_n = \mathrm{tr}\, Q_{\sigma(2)}\, \mathrm{tr}\, Q_{\sigma(3)} \cdots \mathrm{tr}\, Q_{\sigma(n)}\, \mathrm{tr}\, Q_{\sigma(2)}Q_{\sigma(3)}\cdots Q_{\sigma(n)}$$

*for any cyclic permutation* $\sigma$ *on* $\{1,2,\dots,n\}$.

*Proof.* If we show

$$\mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_3 \cdots \mathrm{tr}\, Q_n\, \mathrm{tr}\, Q_1Q_3\cdots Q_n = \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3 \cdots \mathrm{tr}\, Q_n\, \mathrm{tr}\, Q_2Q_3\cdots Q_n,$$

then cyclic permutations of indices yield the desired result. By using (1.4) we rewrite the above equation as

$$\sum_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}} \mathrm{tr}\, Q_1^{2\varepsilon_1} Q_3^{2\varepsilon_3} \cdots Q_n^{2\varepsilon_n} = \sum_{\varepsilon_2,\dots,\varepsilon_n \in \{0,1\}} \mathrm{tr}\, Q_2^{2\varepsilon_2} Q_3^{2\varepsilon_3} \cdots Q_n^{2\varepsilon_n}.$$

By deleting terms common to the both sides of this equation we obtain

$$\sum_{\varepsilon_3,\dots,\varepsilon_n \in \{0,1\}} \mathrm{tr}\, Q_1^2 Q_3^{2\varepsilon_3} \cdots Q_n^{2\varepsilon_n} = \sum_{\varepsilon_3,\dots,\varepsilon_n \in \{0,1\}} \mathrm{tr}\, Q_2^2 Q_3^{2\varepsilon_3} \cdots Q_n^{2\varepsilon_n},$$

which equals

$$\sum_{\varepsilon_3,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_3+\cdots+\varepsilon_n}\, \mathrm{tr}\, P_2P_3^{\varepsilon_3} P_4^{\varepsilon_3+\varepsilon_4} \cdots P_n^{\varepsilon_{n-1}+\varepsilon_n} P_1^{\varepsilon_n+1} = \sum_{\varepsilon_3,\dots,\varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_3+\cdots+\varepsilon_n}\, \mathrm{tr}\, P_2P_3^{1+\varepsilon_3} P_4^{\varepsilon_3+\varepsilon_4} \cdots P_n^{\varepsilon_{n-1}+\varepsilon_n} P_1^{\varepsilon_n}.$$

The last equation is valid by the previous lemma. $\square$

## 2. Parabolic abelian group

**2.1.** For a point $z$ of $\hat{\mathbf{C}}$ we define

$$\Gamma_z = \{A \in SL(2,\mathbf{C}) : \mathrm{tr}\, A \in \{2,-2\} \text{ and } A(z) = z\},$$

which is an abelian group and conjugate in $SL(2, \mathbf{C})$ to

$$\Gamma_\infty = \left\{ \begin{pmatrix} \varepsilon & w \\ 0 & \varepsilon \end{pmatrix} : \varepsilon \in \{-1, 1\}, w \in \mathbf{C} \right\}.$$

There is a group homomorphism $\chi : \Gamma_z \to \{-1, 1\}$ defined by

$$\chi(A) = (1/2) \operatorname{tr} A \quad \text{for } A \in \Gamma_z.$$

Verifying them for the case of $z = \infty$ we obtain the following lemmas and identities.

If $A, B \in \Gamma_z$, then

$$(A - \chi(A)I)(B - \chi(B)I) = O, \tag{2.1}$$

where $O$ is the zero matrix.

LEMMA 2.1. *Let* $A \in \Gamma_z - \{\pm I\}$ *and* $B \in SL(2, \mathbf{C})$. *If* $A$ *and* $B$ *are commutative, then* $B \in \Gamma_z$.

LEMMA 2.2. *Let* $A$ *and* $B \in SL(2, \mathbf{C})$ *be such that* $AB \in \Gamma_z$ *with* $\operatorname{tr} AB = 2$. *Suppose that* $A \notin \Gamma_z$ *and* $\operatorname{tr} A = \operatorname{tr} B$. *Then* $A = B^{-1}$.

LEMMA 2.3. *Let* $A$ *and* $B \in SL(2, \mathbf{C})$ *be such that* $AB$ *is parabolic with* $\operatorname{tr} AB = -2$. *Then* $\operatorname{tr} A + \operatorname{tr} B = 0$ *if and only if* $A$, $B$ *and* $AB$ *have a common fixed point.*

Let $A, B \in SL(2, \mathbf{C})$ be such that $\operatorname{tr} AB = -2$. Then from (1.1) we have

$$\operatorname{tr} A + \operatorname{tr} B = -\operatorname{tr}(A(AB + I)) = -\operatorname{tr}(B(AB + I)). \tag{2.2}$$

LEMMA 2.4. *Let* $P$, $A$ *and* $B \in SL(2, \mathbf{C})$ *and assume that both* $A^{-1}P$ *and* $B^{-1}P$ *are elliptic of order* 2, *then for any* $Q$ *commutative with* $P$ *it holds that*

$$\operatorname{tr} A^{-1}BQ - \operatorname{tr} AB^{-1}Q = 0. \tag{2.3}$$

*Proof.* Since $B^{-1}P = -P^{-1}B$ and $AP^{-1} = -PA^{-1}$ we have

$$\operatorname{tr} A(B^{-1}P)P^{-1}Q = -\operatorname{tr}(AP^{-1})BQP^{-1} = \operatorname{tr} PA^{-1}BQP^{-1}. \qquad \square$$

**2.2.** Let $P_1$ and $P_2$ be two matrices in $SL(2, \mathbf{C})$. We call the pair $\{P_1, P_2\}$ *reducible* if $P_1$ and $P_2$ generate a reducible group and *irreducible* otherwise. The pair $\{P_1, P_2\}$ is reducible if and only if $P_1$ and $P_2$ have a common fixed point (see [1, Definition 1.2.1]). If $\operatorname{tr} P_1 = \operatorname{tr} P_2 = -2$, then the fact that $\{P_1, P_2\}$ is reducible is equivalent to each of the following conditions:

(i) $P_1$ and $P_2$ are commutative.

(ii) $\operatorname{tr} P_1 P_2 = 2$.

We can verify this easily by considering the case where $P_1$ fixes $\infty$.

Lemma 2.5. *Let* $P_1, P_2 \in SL(2, \mathbf{C})$ *with* $\mathrm{tr}\, P_1 = \mathrm{tr}\, P_2 = -2$. *If* $\{P_1, P_2\}$ *is irreducible, then there exists a unique* $Q \in SL(2, \mathbf{C})$ *up to a multiple factor in* $\{-1, 1\}$ *such that*

$$P_1 P_2 = -Q^2. \tag{2.4}$$

*This matrix* $Q$ *satisfies* $\mathrm{tr}\, Q \neq 0$ *and*

$$P_2 = Q^{-1} P_1 Q. \tag{2.5}$$

*Moreover,* $Q^{-1}P_1$ *and* $Q^{-1}P_2$ *are elliptic of order* 2.

*Proof.* By a simultaneous conjugation we can assume that $P_1$ and $P_2$ are of the following forms:

$$P_1 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 \\ \mu & -1 \end{pmatrix},$$

where $\mu \neq 0$. Then we can find a $Q \in SL(2, \mathbf{C})$ such that $P_1 P_2 = -Q^2$. Actually, $Q$ or $-Q$ equals

$$\begin{pmatrix} \sqrt{\mu} & -1/\sqrt{\mu} \\ \sqrt{\mu} & 0 \end{pmatrix}.$$

With this matrix we can easily verify the other properties of $Q$. □

Lemma 2.6. *Let* $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in SL(2, \mathbf{C})$ *be such that* $\mathrm{tr}\, P_1 = \mathrm{tr}\, P_2 = \mathrm{tr}\, P_3 = -2$ *and that* $P_1P_2 = -Q_1^2$, $P_2P_3 = -Q_2^2$, $P_3P_1 = -Q_3^2$. *Then*

$$\begin{aligned} \mathrm{tr}\, P_1P_2P_3 = \frac{1}{2}\, \mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3\, \mathrm{tr}\, Q_1Q_2Q_3 + (\mathrm{tr}\, Q_1)^2 \\ + (\mathrm{tr}\, Q_2)^2 + (\mathrm{tr}\, Q_3)^2 - 2 \end{aligned} \tag{2.6}$$

*and*

$$\begin{aligned} \mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_1Q_2 &= \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3\, \mathrm{tr}\, Q_2Q_3 = \mathrm{tr}\, Q_3\, \mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_3Q_1 \\ &= -\mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3\, \mathrm{tr}\, Q_1Q_2Q_3 - (\mathrm{tr}\, Q_1)^2 \\ &\quad - (\mathrm{tr}\, Q_2)^2 - (\mathrm{tr}\, Q_3)^2. \end{aligned} \tag{2.7}$$

*Proof.* By (1.1) and (1.4) we have

$$\begin{aligned} \mathrm{tr}\, Q_1\, \mathrm{tr}\, Q_2\, \mathrm{tr}\, Q_3\, \mathrm{tr}\, Q_1Q_2Q_3 &= \mathrm{tr}\, Q_1^2Q_2^2Q_3^2 + \mathrm{tr}\, Q_1^2Q_2^2 + \mathrm{tr}\, Q_2^2Q_3^2 \\ &\quad + \mathrm{tr}\, Q_3^2Q_1^2 + \mathrm{tr}\, Q_1^2 + \mathrm{tr}\, Q_2^2 + \mathrm{tr}\, Q_3^2 + \mathrm{tr}\, I \\ &= -\mathrm{tr}\, P_1^2P_2^2P_3^2 + \mathrm{tr}\, P_1P_2^2P_3 + \mathrm{tr}\, P_2P_3^2P_1 + \mathrm{tr}\, P_1^2P_2P_3 \\ &\quad - \mathrm{tr}\, P_1P_2 - \mathrm{tr}\, P_2P_3 - \mathrm{tr}\, P_3P_1 + 2 \\ &= 2\, \mathrm{tr}\, P_1P_2P_3 + 2\, \mathrm{tr}\, P_1P_2 + 2\, \mathrm{tr}\, P_2P_3 + 2\, \mathrm{tr}\, P_3P_1 - 8. \end{aligned}$$

Since $\operatorname{tr} Q_i^2 = (\operatorname{tr} Q_i)^2 - 2$ we obtain (2.6).

$$\begin{aligned}\operatorname{tr} Q_1 \operatorname{tr} Q_2 \operatorname{tr} Q_1Q_2 &= \operatorname{tr} Q_1^2Q_2^2 + \operatorname{tr} Q_1^2 + \operatorname{tr} Q_2^2 + 2 \\ &= -2 \operatorname{tr} P_1P_2P_3 + (\operatorname{tr} Q_1)^2 + (\operatorname{tr} Q_2)^2 + (\operatorname{tr} Q_3)^2 - 4.\end{aligned}$$

This, together with (2.6) and Lemma 1.4, yields (2.7). □

## 3. Proof of the main theorem

**3.1.** Let $Q_1, Q_2, \dots, Q_n \in SL(2,\mathbf{C})$ be such that $Q_1Q_2\cdots Q_n$ is commutative with a parabolic matrix in $SL(2,\mathbf{C})$. Then by Lemma 2.1 we have

$$\operatorname{tr} Q_1Q_2\cdots Q_n \in \{-2, 2\}.$$

Definition 3.1. *The ordered tuple $(Q_1, Q_2, \dots, Q_n)$ is called a $(-)$-sytem if* $\operatorname{tr} Q_1Q_2\cdots Q_n = -2$ *and a $(+)$-system if* $\operatorname{tr} Q_1Q_2\cdots Q_n = 2$.

Let $P_1, P_2, \dots, P_n$ be parabolic matrices in $SL(2,\mathbf{C})$ such that $\operatorname{tr} P_1 = \operatorname{tr} P_2 = \cdots = \operatorname{tr} P_n = -2$. Since $\operatorname{tr} P_i = \operatorname{tr} P_{i+1}$ for $i = 1, 2, \dots n$, where $P_{n+1} = P_1$ and since $P_i \neq \pm I$, there exist a matrix $Q_i$ such that $P_{i+1} = Q_i^{-1}P_iQ_i$. It holds that

$$\begin{aligned}P_1 &= Q_n^{-1}P_nQ_n \\ &= (Q_{n-1}Q_n)^{-1}P_{n-1}(Q_{n-1}Q_n) = \cdots = (Q_1\cdots Q_n)^{-1}P_1(Q_1\cdots Q_n).\end{aligned}$$

Thus $Q_1Q_2\cdots Q_n$ and $P_1$ are commutative and hence $(Q_1, Q_2, \dots, Q_n)$ is either a $(-)$-system or a $(+)$-system.

Let $P_1, P_2, P_3, P_4 \in SL(2,\mathbf{C})$ be such that $\operatorname{tr} P_1 = \operatorname{tr} P_2 = \operatorname{tr} P_3 = \operatorname{tr} P_4 = -2$. We assume that any two of $P_1$, $P_2$, $P_3$, $P_4$ generate an irreducible group. Then by Lemma 2.5 we can find $Q_1, Q_2, \dots, Q_6 \in SL(2,\mathbf{C})$ satisfying the following conditions.

$$\begin{aligned}&P_1P_2 = -Q_1^2, \quad P_2P_3 = -Q_2^2, \quad P_3P_4 = -Q_3^2, \\ &P_4P_1 = -Q_4^2, \quad P_3P_1 = -Q_5^2, \quad P_4P_2 = -Q_6^2\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&P_2 = Q_1^{-1}P_1Q_1, \quad P_3 = Q_2^{-1}P_2Q_2, \quad P_4 = Q_3^{-1}P_3Q_3, \\ &P_1 = Q_4^{-1}P_4Q_4, \quad P_1 = Q_5^{-1}P_3Q_5, \quad P_2 = Q_6^{-1}P_4Q_6\end{aligned} \tag{3.2}$$

These $Q_i$ are uniquely determined up to a multiple factor in $\{-1, +1\}$. Let $\tilde{Q}_5 = P_1Q_5P_1^{-1}$ and $\tilde{Q}_6 = P_2Q_6P_2^{-1}$. Then $\operatorname{tr} Q_5 = \operatorname{tr} \tilde{Q}_5$ and $\operatorname{tr} Q_6 = \operatorname{tr} \tilde{Q}_6$. Since $\tilde{Q}_5^2 = -P_1P_3$ and $\tilde{Q}_6^2 = -P_2P_4$, Lemma 2.5 yields

$$\tilde{Q}_5^{-1}P_1\tilde{Q}_5 = P_3 \quad \text{and} \quad \tilde{Q}_6^{-1}P_2\tilde{Q}_6 = P_4.$$

Lemma 3.1. *Assume that* $(Q_1, Q_2, Q_5)$ *and* $(Q_2, Q_3, Q_6)$ *are* $(-)$*-systems. Then each of* $(Q_3, Q_4, \tilde{Q}_5)$ *and* $(Q_4, Q_1, \tilde{Q}_6)$ *is a* $(-)$*-system if and only if* $(Q_1, Q_2, Q_3, Q_4)$ *is a* $(-)$*-system.*

*Proof.* Since $Q_5^{-1}Q_3Q_4$ commutes with $P_1$ and since $P_1^{-1}Q_5$ is elliptic of order 2, we obtain

$$\begin{aligned}\chi(\tilde{Q}_5Q_3Q_4) &= \chi(P_1^2(P_1^{-1}Q_5)^2Q_5^{-1}Q_3Q_4)\\ &= \chi(P_1)^2\chi(-I)\chi(Q_5^{-1}Q_3Q_4) = -\chi(Q_5^{-1}Q_3Q_4).\end{aligned}$$

Therefore, $\operatorname{tr} \tilde{Q}_5Q_3Q_4 = -2$ if and only if $\operatorname{tr} Q_5^{-1}Q_3Q_4 = 2$. Since

$$\operatorname{tr} Q_1Q_2Q_3Q_4 = 2\chi(Q_1Q_2Q_5)\chi(Q_5^{-1}Q_3Q_4),$$

$\operatorname{tr} Q_1Q_2Q_3Q_4 = -2$ if and only if $\operatorname{tr} Q_5^{-1}Q_3Q_4 = 2$. The proof for $(Q_4, Q_1, \tilde{Q}_6)$ is similar. □

**3.2.** Now we prove Theorem 0.1. Let $\lambda_1 = \operatorname{tr} Q_1$, $\lambda_2 = \operatorname{tr} Q_2$, $\lambda_3 = \operatorname{tr} Q_3$, $\lambda_4 = \operatorname{tr} Q_4$, $\lambda_5 = \operatorname{tr} Q_5$ and $\lambda_6 = \operatorname{tr} Q_6$. By Lemma 3.1 we need to show the equation

$$\lambda_1\lambda_3 + \lambda_2\lambda_4 = \lambda_5\lambda_6 \tag{3.3}$$

when $(Q_1, Q_2, Q_5)$, $(Q_3, Q_4, \tilde{Q}_5)$, $(Q_2, Q_3, Q_6)$ and $(Q_4, Q_1, \tilde{Q}_6)$ are all $(-)$-systems.

Our proof proceeds as in the following way: In 3.3 we prove (3.5) below, which is the same equation as

$$(\lambda_1\lambda_3 + \lambda_2\lambda_4 - \lambda_5\lambda_6)(\lambda_1\lambda_3 + \lambda_2\lambda_4 + \lambda_5\lambda_6) = 0.$$

In 3.4, we introduce a value $\Delta$ defined by $\lambda_1 + \operatorname{tr} Q_2Q_3Q_4 = \lambda_1\Delta$ and prove (3.3) provided $\Delta \neq 0$. The proof is an application of Lemma 1.2. In 3.5 we obtain an explicit expression of $\Delta$ by using the facts that $(Q_1, Q_2, Q_5)$, $(Q_3, Q_4, \tilde{Q}_5)$, $(Q_2, Q_3, Q_6)$ and $(Q_4, Q_1, \tilde{Q}_6)$ are $(-)$-systems. Finally, in 3.6 we show that two equations $\lambda_1\lambda_3 + \lambda_2\lambda_4 + \lambda_5\lambda_6 = 0$ and $\Delta = 0$ are not compatible. In the calculations of trace identities, as illustrated in the proof, one often encounters surplus and cumbersome equations such as $\lambda_1\lambda_3 + \lambda_2\lambda_4 + \lambda_5\lambda_6 = 0$.

**3.3.** From (1.3) we obtain

$$\begin{aligned}2 \operatorname{tr} P_1P_2P_3P_4 = &-2 \operatorname{tr} P_2P_3P_4 - 2 \operatorname{tr} P_1P_3P_4 - 2 \operatorname{tr} P_1P_2P_4 - 2 \operatorname{tr} P_1P_2P_3\\ &+ \operatorname{tr} P_1P_2 \operatorname{tr} P_3P_4 - \operatorname{tr} P_1P_3 \operatorname{tr} P_2P_4 + \operatorname{tr} P_4P_1 \operatorname{tr} P_2P_3\\ &- 4 \operatorname{tr} P_3P_4 - 4 \operatorname{tr} P_4P_1 - 4 \operatorname{tr} P_1P_2 - 4 \operatorname{tr} P_2P_3 + 16\end{aligned} \tag{3.4}$$

From (1.4), (3.1) and (3.4) we obtain

$$\begin{aligned}
&\text{tr } Q_1 \text{ tr } Q_2 \text{ tr } Q_3 \text{ tr } Q_4 \text{ tr } Q_1Q_2Q_3Q_4\\
&\quad= \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{0,1\}} Q_1^{2\varepsilon_1}Q_2^{2\varepsilon_2}Q_3^{2\varepsilon_3}Q_4^{2\varepsilon_4}\\
&\quad= 2 \text{ tr } P_1P_2P_3P_4 + 2(\text{tr } P_1P_2P_3 + \text{tr } P_1P_2P_4 + \text{tr } P_1P_3P_4 + \text{tr } P_2P_3P_4)\\
&\qquad + 2(\text{tr } P_1P_2 + \text{tr } P_1P_3 + \text{tr } P_1P_4 + \text{tr } P_2P_3 + \text{tr } P_2P_4 + \text{tr } P_3P_4) - 12\\
&\quad= \text{tr } P_1P_2 \text{ tr } P_3P_4 - \text{tr } P_1P_3 \text{ tr } P_2P_4 + \text{tr } P_1P_4 \text{ tr } P_2P_3\\
&\qquad + 2(\text{tr } P_1P_3 + \text{tr } P_2P_4 - \text{tr } P_1P_2 - \text{tr } P_2P_3 - \text{tr } P_3P_4 - \text{tr } P_4P_1) + 4.
\end{aligned}$$

Since $\text{tr } Q_1Q_2Q_3Q_4 = -2$, this equation, together with (3.1), yields

$$\lambda_5^2\lambda_6^2 = (\lambda_1\lambda_3 + \lambda_2\lambda_4)^2. \tag{3.5}$$

**3.4.** By Lemma 1.4

$$\text{tr } Q_1 \text{ tr } Q_2 \text{ tr } Q_3 \text{ tr } Q_4 + \text{tr } Q_2 \text{ tr } Q_3 \text{ tr } Q_4 \text{ tr } Q_2Q_3Q_4$$

is invariant under any cyclic permutation of the indices $\{1,2,3,4\}$. Therefore there exists a $\Delta$ such that

$$\begin{aligned}
&\text{tr } Q_1 + \text{tr } Q_2Q_3Q_4 = \lambda_1\Delta, \quad \text{tr } Q_2 + \text{tr } Q_1Q_3Q_4 = \lambda_2\Delta,\\
&\text{tr } Q_3 + \text{tr } Q_1Q_2Q_4 = \lambda_3\Delta, \quad \text{tr } Q_4 + \text{tr } Q_1Q_2Q_3 = \lambda_4\Delta.
\end{aligned} \tag{3.6}$$

Next we shall show the following identities

$$\text{tr } Q_1Q_2 + \text{tr } Q_3Q_4 = \lambda_5\Delta, \quad \text{tr } Q_2Q_3 + \text{tr } Q_4Q_1 = \lambda_6\Delta. \tag{3.7}$$

By the proof of Lemma 3.1 we know that $\text{tr } Q_5^{-1}Q_3Q_4 = 2$. Then from (2.1)

$$\begin{aligned}
O &= (Q_1Q_2Q_5 + I)(-I + Q_5^{-1}Q_3Q_4)\\
&= -Q_1Q_2Q_5 + Q_1Q_2Q_3Q_4 - I + Q_5^{-1}Q_3Q_4.
\end{aligned} \tag{3.8}$$

Since $\text{tr } Q_1Q_2Q_5 = -2$, $2I + Q_1Q_2Q_5 = -(Q_1Q_2Q_5)^{-1}$ by the Cayley-Hamilton theorem. Then (1.1), (2.2) and (3.8) yield

$$\begin{aligned}
&-\text{tr } Q_5(\text{tr } Q_1 + \text{tr } Q_2Q_3Q_4)\\
&\quad= \text{tr } Q_5 \text{ tr } Q_1(Q_1Q_2Q_3Q_4 + I)\\
&\quad= \text{tr } Q_5 \text{ tr } Q_1(Q_1Q_2Q_5 + 2I - Q_5^{-1}Q_3Q_4)\\
&\quad= -\text{tr } Q_5 \text{ tr } Q_1((Q_1Q_2Q_5)^{-1} + Q_5^{-1}Q_3Q_4)\\
&\quad= -\text{tr } Q_5 \text{ tr } Q_1 \text{ tr}(Q_1Q_2Q_5 + Q_5^{-1}Q_3Q_4)\\
&\qquad + \text{tr } Q_5 \text{ tr}(Q_5^{-1}Q_2^{-1}Q_1^{-2} + Q_1^{-1}Q_5^{-1}Q_3Q_4)\\
&\quad= -\text{tr } Q_1(\text{tr } Q_1Q_2 + \text{tr } Q_3Q_4) - \text{tr } Q_1 \text{ tr}(Q_1Q_2Q_5^2 + Q_5^{-2}Q_3Q_4)\\
&\qquad + \text{tr } Q_5 \text{ tr}(Q_1^2Q_2Q_5 + Q_1^{-1}Q_5^{-1}Q_3Q_4)
\end{aligned}$$

$$= -\mathrm{tr}\ Q_1(\mathrm{tr}\ Q_1Q_2 + \mathrm{tr}\ Q_3Q_4) + (\mathrm{tr}\ Q_5\ \mathrm{tr}\ Q_1^2Q_2Q_5 - \mathrm{tr}\ Q_1\ \mathrm{tr}\ Q_1Q_2Q_5^2)$$
$$+ (\mathrm{tr}\ Q_5\ \mathrm{tr}\ Q_1^{-1}Q_5^{-1}Q_3Q_4 - \mathrm{tr}\ Q_1\ \mathrm{tr}\ Q_5^{-2}Q_3Q_4)$$
$$= -\mathrm{tr}\ Q_1(\mathrm{tr}\ Q_1Q_2 + \mathrm{tr}\ Q_3Q_4) - (\mathrm{tr}\ Q_5\ \mathrm{tr}\ Q_2Q_5 - \mathrm{tr}\ Q_1\ \mathrm{tr}\ Q_1Q_2)$$
$$+ (\mathrm{tr}\ Q_1^{-1}Q_3Q_4 - \mathrm{tr}\ Q_1Q_5^{-2}Q_3Q_4)$$

We treat the last two terms. Since $P_3 = Q_2^{-1}P_2Q_2$, (1.1) yields

$$\mathrm{tr}\ Q_5\ \mathrm{tr}\ Q_2Q_5 - \mathrm{tr}\ Q_1\ \mathrm{tr}\ Q_1Q_2$$
$$= \mathrm{tr}\ Q_2Q_5^2 - \mathrm{tr}\ Q_1^2Q_2 = -\mathrm{tr}\ Q_2P_3P_1 + \mathrm{tr}\ P_1P_2Q_2 = 0$$

From Lemma 2.4 we have

$$\mathrm{tr}\ Q_1^{-1}Q_5(Q_5^{-1}Q_3Q_4) - \mathrm{tr}\ Q_1Q_5^{-1}(Q_5^{-1}Q_3Q_4) = 0,$$

for $Q_1P_1^{-1}$ and $Q_5^{-1}P_1$ are elliptic of order 2 and $P_1$ and $Q_5^{-1}Q_3Q_4$ are commutative. So we conclude that $\mathrm{tr}\ Q_5(\mathrm{tr}\ Q_1 + \mathrm{tr}\ Q_2Q_3Q_4) = \mathrm{tr}\ Q_1(\mathrm{tr}\ Q_1Q_2 + \mathrm{tr}\ Q_3Q_4)$ and also the first equation in (3.7). The same calculation(or permuting the indices in the calculation above) yields the second one. By using Lemma 1.2 and (3.6) and (3.7) we can conclude (3.3) provided that $\Delta$ is not zero.

**3.5.** In this subsection we find the explicit expressions of $\Delta$. Applying (2.7) in Lemma 2.6 to the $(-)$-system $(Q_1, Q_2, Q_5)$ and $(Q_3, Q_4, \tilde{Q}_5)$, we obtain

$$\mathrm{tr}\ Q_1Q_2 = 2\lambda_5 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_5^2}{\lambda_1\lambda_2}, \quad \mathrm{tr}\ Q_3Q_4 = 2\lambda_5 - \frac{\lambda_3^2 + \lambda_4^2 + \lambda_5^2}{\lambda_3\lambda_4}.$$

Thus, by using the first equation of (3.7) we obtain

$$\Delta = 4 - \frac{(\lambda_1\lambda_3 + \lambda_2\lambda_4)(\lambda_2\lambda_3 + \lambda_1\lambda_4) + \lambda_5^2(\lambda_1\lambda_2 + \lambda_3\lambda_4)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}. \tag{3.9}$$

Since $(Q_2, Q_3, Q_6)$ and $(Q_4, Q_1, \tilde{Q}_6)$ are also $(-)$-systems, by repeating the same argument and using the second equation of (3.7) we obtain also

$$\Delta = 4 - \frac{(\lambda_1\lambda_3 + \lambda_2\lambda_4)(\lambda_1\lambda_2 + \lambda_3\lambda_4) + \lambda_6^2(\lambda_2\lambda_3 + \lambda_1\lambda_4)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6}. \tag{3.10}$$

**3.6.** The difference of the expressions (3.9) and (3.10) of $\Delta$ yields

$$(\lambda_5\lambda_6 - \lambda_1\lambda_3 - \lambda_2\lambda_4)[(\lambda_1\lambda_2 + \lambda_3\lambda_4)\lambda_5 - (\lambda_2\lambda_3 + \lambda_1\lambda_4)\lambda_6] = 0.$$

So, if (3.3) does not hold, by the results in 3.3 and 3.4 we obtain the following three equations.

$$\lambda_5\lambda_6 + (\lambda_1\lambda_3 + \lambda_2\lambda_4) = 0 \tag{3.11}$$

$$(\lambda_1\lambda_2 + \lambda_3\lambda_4)\lambda_5 - (\lambda_2\lambda_3 + \lambda_1\lambda_4)\lambda_6 = 0 \tag{3.12}$$

$$\Delta = 0 \tag{3.13}$$

From (3.11) and (3.12) we have

$$(\lambda_1\lambda_3+\lambda_2\lambda_4)(\lambda_2\lambda_3+\lambda_1\lambda_4)+\lambda_5^2(\lambda_1\lambda_2+\lambda_3\lambda_4)$$
$$=(\lambda_2\lambda_3+\lambda_1\lambda_4)[(\lambda_1\lambda_3+\lambda_2\lambda_4)+\lambda_5\lambda_6]=0.$$

Then (3.9) yields $\Delta=4$, which contradicts (3.13). Now we complete the proof of (3.3) and hence Theorem 0.1.

## 4. Degenerate cases

In this section we extend Theorem 0.1 to the full generality so that it is valid for the case where $P_1$, $P_2$, $P_3$ and $P_4$ include reducible pairs.

**4.1.** Let $P_1,P_2\in SL(2,\mathbf{C})$ be such that $\operatorname{tr}P_1=\operatorname{tr}P_2=-2$. If $\{P_1,P_2\}$ is an irreducible pair, Lemma 2.5 yields a matrix $Q_1$ satisfying $P_1P_2=-Q_1^2$. Then (1.1) yields $(\operatorname{tr}Q_1)^2=2-\operatorname{tr}P_1P_2$. If $\{P_1,P_2\}$ is reducible, then $\operatorname{tr}P_1P_2=2$. So it seems natural to define the value corresponding $\operatorname{tr}Q_1$ to be zero for this case.

**4.2.** Let $P_1,P_2\in SL(2,\mathbf{C})$ be parabolic with $\operatorname{tr}P_1=\operatorname{tr}P_2=-2$. Even for the case where $\{P_1,P_2\}$ is reducible, we can find matrices $Q_1$ satisfying $P_2=Q_1^{-1}P_1Q_1$ (but not $P_1P_2=-Q_1^2$). Such matrices form essentially a one-parameter family: If

$$P_1=\begin{pmatrix}-1&-1\\0&-1\end{pmatrix},\quad P_2=\begin{pmatrix}-1&-\mu\\0&-1\end{pmatrix},$$

where $\mu$ is a nonzero complex number, then

$$Q_1=\pm\begin{pmatrix}1/\sqrt{\mu}&\xi\\0&\sqrt{\mu}\end{pmatrix},\tag{4.1}$$

with an arbitrary complex number $\xi$.

**4.3.** Let $P_1$, $P_2$, $P_3$ and $P_4\in SL(2,\mathbf{C})$ be parabolic with $\operatorname{tr}P_i=-2$ for $i=1,2,3,4$. We choose matrices $Q=Q_{[P_i,P_j]}\in SL(2,\mathbf{C})$ satisfying $P_j=Q^{-1}P_iQ$. If $\{P_i,P_j\}$ is irreducible, we impose the condition $P_iP_j=-Q^2$ on $Q$ too. Let

$$Q_1=Q_{[P_1,P_2]},\quad Q_2=Q_{[P_2,P_3]},\quad Q_3=Q_{[P_3,P_4]},$$
$$Q_4=Q_{[P_4,P_1]},\quad Q_5=Q_{[P_3,P_1]},\quad Q_6=Q_{[P_4,P_2]}.$$

If the indices $i$, $j$, $k$ are such that $Q_k=Q_{[P_i,P_j]}$, then we define $\lambda_k=\operatorname{tr}Q_k$ if $\{P_i,P_j\}$ is irreducible and $\lambda_k=0$ otherwise.

Theorem 4.1. *Under the assumption that*

$$(Q_1,Q_2,Q_5),\quad (Q_2,Q_3,Q_6)\quad \textit{and}\quad (Q_1,Q_2,Q_3,Q_4)$$

*are* $(-)$*-systems, the same equation as in* (3.3) *holds.*

**4.4.** We prove Theorem 4.1 for the case where there are distinct indices $i, j \in \{1, 2, 3, 4\}$ such that $\{P_i, P_j\}$ is reducible but $\{P_i, P_k\}$ is irreducible for any $k \neq i, j$. We consider subcases.

**4.4.1.** We give a proof for the case $\{i, j\} = \{2, 3\}$. Since $P_1 = Q_4^{-1} P_4 Q_4$ and $P_3 = Q_2^{-1} P_2 Q_2$, we obtain

$$\begin{aligned} \operatorname{tr} Q_3 \operatorname{tr} Q_2 Q_3 Q_4 &= \operatorname{tr} Q_2 Q_3^2 Q_4 + \operatorname{tr} Q_2 Q_4 \\ &= -\operatorname{tr} Q_2 P_3 P_4 Q_4 + \operatorname{tr} Q_2 Q_4 \\ &= -\operatorname{tr} P_2 Q_2 Q_4 P_1 + \operatorname{tr} Q_2 Q_4 \\ &= \operatorname{tr} Q_1 \operatorname{tr} Q_4 Q_1 Q_2. \end{aligned}$$

Thus there exists $\Delta$ such that

$$\operatorname{tr} Q_1 + \operatorname{tr} Q_2 Q_3 Q_4 = \lambda_1 \Delta, \quad \operatorname{tr} Q_3 + \operatorname{tr} Q_1 Q_2 Q_4 = \lambda_3 \Delta. \tag{4.2}$$

We can show

$$\operatorname{tr} Q_1 Q_2 + Q_3 Q_4 = \lambda_5 \Delta, \quad \operatorname{tr} Q_2 Q_3 + \operatorname{tr} Q_4 Q_1 = \lambda_6 \Delta \tag{4.3}$$

by the same calculation as in Subsection 3.4. With the facts that $Q_1 P_1^{-1}$ and $Q_5^{-1} P_1$ are elliptic of order 2 and that $P_3 = Q_2^{-1} P_2 Q_2$, we obtain

$$\operatorname{tr} Q_5 (\operatorname{tr} Q_1 + \operatorname{tr} Q_2 Q_3 Q_4) = \operatorname{tr} Q_1 (\operatorname{tr} Q_1 Q_2 + Q_3 Q_4),$$

and hence the first equation in (4.3). Likewise, since $Q_4 P_4^{-1}$ and $\tilde{Q}_6^{-1} P_4$ are elliptic of order 2, where $\tilde{Q}_6 = P_2 Q_6 P_2^{-1}$, and since $P_2 = Q_1^{-1} P_1 Q_1$, we obtain

$$\operatorname{tr} \tilde{Q}_6 (\operatorname{tr} Q_4 + \operatorname{tr} Q_1 Q_2 Q_3) = \operatorname{tr} Q_4 (\operatorname{tr} Q_4 Q_1 + Q_2 Q_3),$$

and also the second one in (4.3).

Since $P_3 = Q_2^{-1} P_2 Q_2$ and $P_2 = (Q_3 Q_4 Q_1)^{-1} P_3 (Q_3 Q_4 Q_1)$, $Q_2$ and $Q_3 Q_4 Q_1$ fix the common fixed point of $P_2$ and $P_3$. Hence by Lemma 2.3,

$$\operatorname{tr} Q_2 + \operatorname{tr} Q_3 Q_4 Q_1 = 0. \tag{4.4}$$

We suppose that $Q_2$ is of the form in (4.1). Since $P_2 = Q_1^{-1} P_1 Q_1$, $Q_1$ sends the fixed point $\infty$ of $P_2$ to that of $P_1$. Hence $Q_1(\infty) \neq \infty$ (that is, the (2,1)-entry of $Q_1$ is nonzero). Thus $\operatorname{tr} Q_1 Q_2$ is of the form $a\xi + b$ with complex constants $a \neq 0, b$. We can choose $\xi$ so that $\operatorname{tr} Q_1 Q_2 + \operatorname{tr} Q_3 Q_4 \neq 0$ and hence $\Delta \neq 0$. Now Lemma 1.2, together with (4.2), (4.3) and (4.4), yields (3.3), which is

$$\lambda_5 \lambda_6 = \lambda_1 \lambda_3$$

for the present case.

**4.4.2.** Next we treat the case where $\{P_1, P_3\}$ is reducible. We consider the ordered tuple $(P_2, P_3, P_1, P_4)$ instead of $(P_1, P_2, P_3, P_4)$. We note that

$$P_3 = Q_2^{-1}P_2Q_2, \quad P_1 = Q_5^{-1}P_3Q_5 \quad P_4 = (-P_1Q_4P_1^{-1})^{-1}P_1(-P_1Q_4P_1^{-1})$$
$$P_2 = Q_6^{-1}P_4Q_6, \quad P_2 = Q_1^{-1}P_1Q_1 \quad P_3 = (P_4Q_3P_4^{-1})^{-1}P_4(P_4Q_3P_4^{-1})$$

and

$$P_2P_3 = -Q_2^2, \quad P_1P_4 = -(-P_1Q_4P_1^{-1})^2,$$
$$P_1P_2 = -Q_1^2, \quad P_4P_3 = -(P_4Q_3P_4^{-1})^2.$$

(It may holds that $P_4P_2 = -Q_6^2$.) The reason for the choice of $-P_1Q_4P_1^{-1}$ is that, as shown below, $(Q_2, Q_5, -P_1Q_4P_1^{-1}, Q_6)$ is a $(-)$-system. Then, since $(Q_6, Q_2, Q_3)$ is a $(-)$-system, so is $(Q_5, -P_1Q_4P_1^{-1}, P_4Q_3P_4^{-1})$. Since $(Q_2, Q_5, Q_1)$ is also a $(-)$-system, from the result of 4.4.1 we obtain

$$\operatorname{tr} Q_1 \operatorname{tr}(P_4Q_3P_4^{-1}) = \operatorname{tr} Q_2(-\operatorname{tr} P_1Q_4P_1^{-1}),$$

which is the desired equation

$$\lambda_1\lambda_3 + \lambda_2\lambda_4 = 0.$$

To show that $(Q_2, Q_5, -P_1Q_4P_1^{-1}, Q_6)$ is a $(-)$-system, note that $Q_4P_1^{-1}$ is elliptic of order 2 and that $P_1$ and $Q_4^{-1}Q_6Q_2Q_5$ commute. Then

$$\operatorname{tr} Q_2Q_5(-P_1Q_4P_1^{-1})Q_6 = 2\chi(Q_4^{-1}Q_6Q_2Q_5)\cdot\chi(P_1^2) = \operatorname{tr} Q_4^{-1}Q_6Q_2Q_5.$$

Moreover $Q_6Q_2Q_3$ and $Q_3^{-1}Q_5Q_4^{-1}$ commute $P_4$. As shown in the proof of Lemma 3.1, $\operatorname{tr} Q_5^{-1}Q_3Q_4 = 2$. Therefore we have

$$\begin{aligned}\operatorname{tr} Q_4^{-1}Q_6Q_2Q_5 &= \operatorname{tr}(Q_6Q_2Q_3)(Q_3^{-1}Q_5Q_4^{-1})\\ &= 2\chi(Q_2Q_3Q_6)\chi(Q_3^{-1}Q_5Q_4^{-1})\\ &= 2\cdot(-1)\cdot(+1) = -2.\end{aligned}$$

Now we conclude the proof. (Other cases can be proved in a similar way after a cyclic permutation of indices $\{1,2,3,4\}$.)

**4.5.** Let three of $P_1$, $P_2$, $P_3$ and $P_4$ generate a reducible group. By a cyclic permutation of indices, we can assume that those three are $P_1$, $P_2$ and $P_3$. Then $\lambda_1 = \lambda_2 = \lambda_5 = 0$ and so (3.3) is trivial. Now we complete the proof of Theorem 4.1.

## 5. An example

**5.1.** *The four-times puntured sphere.* Let $F$ be a two-dimensional sphere and $P = \{x_1, x_2, x_3, x_4\}$ a set of four distinct points of $F$. Let $F' = F - P$. The fundamental group of $F'$ is isomorphic to the group $G$ with the presentation

$$\langle c_1, c_2, c_3, c_4 : c_1c_2c_3c_4 = 1\rangle.$$

We define $\tilde{\mathscr{R}}$ to be the set of all faithful representations $\rho$ of $G$ into $SL(2,\mathbf{C})$ satisfying $\operatorname{tr}\rho(c_i) = -2$ for $i = 1,2,3,4$ and $\mathscr{R}(= R_{0,4})$ to be the set of conjugacy classes $[\rho]$ of $\rho$ in $\tilde{\mathscr{R}}$. Since $\rho \in \tilde{\mathscr{R}}$ is determined by the matrices $P_i = \rho(c_i)$, $i = 1,2,3$, we shall identify $\rho$ with $(P_1, P_2, P_3)$ and $[\rho]$ with the (simultaneous) conjugacy class of $(P_1, P_2, P_3)$.

**5.2.** Our first purpose of this section is to introduce a coordinate-system for $\mathscr{R}$. Let $\rho = (P_1, P_2, P_3) \in \tilde{\mathscr{R}}$. Since $\rho$ is faithful, all pairs in $P_1$, $P_2$, $P_3$ and $P_4 = (P_1P_2P_3)^{-1}$ are irreducible. We choose $Q_1, \dots, Q_6 \in SL(2,\mathbf{C})$ so that (3.1) holds. Since $P_1P_2P_3P_4 = I$, we have

$$Q_3^2 = -P_3P_4 = -(P_1P_2)^{-1} = Q_1^{-2} \quad \text{and} \quad Q_4^2 = -P_4P_1 = -(P_2P_3)^{-1} = Q_2^{-2}.$$

Thus, from Lemma 2.5, $Q_3 = Q_1^{-1}$ or $Q_3 = -Q_1^{-1}$ and $Q_4 = Q_2^{-1}$ or $Q_4 = -Q_2^{-1}$. It holds that

$$\operatorname{tr} Q_1Q_2Q_1^{-1}Q_2^{-1} = -2. \tag{5.1}$$

For, if $\operatorname{tr} Q_1Q_2Q_1^{-1}Q_2^{-1} = 2$, then $Q_1$ and $Q_2$ generate a reducible group and hence $Q_1$ and $Q_2$ have a common fixed point ([1, lemma 1.2.3]). Since $Q_1Q_2Q_1^{-1}Q_2^{-1}$ and $P_1$ commute, this fixed point is also fixed by $P_1$ and then by $P_2 = Q_1^{-1}P_1Q_1$. This contradicts that $\{P_1, P_2\}$ is an irreducible pair. Now (5.1) holds and it implies that $(Q_1, Q_2, Q_1^{-1}, Q_2^{-1})$ is a $(-)$-system. We redefine $Q_3$, $Q_4$ so that $Q_3 = Q_1^{-1}$ and $Q_4 = Q_2^{-1}$.

Suppose that $(Q_1, Q_2, Q_5)$ and $(Q_2, Q_1^{-1}, Q_6)$ are $(-)$-systems. (Otherwise we need only to replace $Q_5$ by $-Q_5$ and/or $Q_6$ by $-Q_6$.) From (3.6) we have $\operatorname{tr} Q_1 + \operatorname{tr} Q_2Q_1^{-1}Q_2^{-1} = (\operatorname{tr} Q_1)\Delta$. Thus $\Delta = 2$. Then (3.7) yields

$$\operatorname{tr} Q_5 = \operatorname{tr} Q_1^{-1}Q_2^{-1} \quad \text{and} \quad \operatorname{tr} Q_6 = \operatorname{tr} Q_2^{-1}Q_1. \tag{5.2}$$

Both $Q_1Q_2Q_5$ and $Q_1Q_2Q_1^{-1}Q_2^{-1}$ are commutative with $P_1$ and

$$\operatorname{tr} Q_5^{-1}Q_1^{-1}Q_2^{-1} = 2\chi((Q_1Q_2Q_5)^{-1})\chi(Q_1Q_2Q_1^{-1}Q_2^{-1}) = 2.$$

Likewise $Q_2Q_1^{-1}Q_6$ and $Q_2Q_1^{-1}Q_2^{-1}Q_1$ commute with $P_2$ and $\operatorname{tr} Q_6^{-1}Q_2^{-1}Q_1 = 2$. So Lemma 2.2, together with (5.2), yields

$$Q_5 = Q_1^{-1}Q_2^{-1} \quad \text{and} \quad Q_6 = Q_2^{-1}Q_1.$$

Let $\lambda_1 = \operatorname{tr} Q_1$, $\lambda_2 = \operatorname{tr} Q_2$ and $\lambda_3 = \operatorname{tr} Q_5$. Since $\operatorname{tr} P_1P_2P_3 = \operatorname{tr} P_4 = -2$ and $\operatorname{tr} Q_1Q_2Q_5 = -2$, (2.6) yields the following identity.

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2\lambda_3 = 0 \tag{5.3}$$

**5.3.** Let $\mathscr{P}$ be the set of all pairs $(Q_1, Q_2)$ of matrices of $SL(2,\mathbf{C})$ such that $(Q_1, Q_2, Q_1^{-1}Q_2^{-1})$ is a $(-)$-system and such that there exist $P_1$, $P_2$ and $P_3 \in SL(2,\mathbf{C})$ with $\operatorname{tr} P_1 = \operatorname{tr} P_2 = \operatorname{tr} P_3 = \operatorname{tr}(P_1P_2P_3) = -2$ satisfying

$$P_1P_2 = -Q_1^2, \quad P_2P_3 = -Q_2^2, \quad P_3P_1 = -(Q_1^{-1}Q_2^{-1})^2. \tag{5.4}$$

Let $\pi : \mathscr{P} \to \tilde{\mathscr{R}}$ be the mapping sending $(Q_1, Q_2)$ to $(P_1, P_2, P_3)$. If $\pi(Q_1, Q_2) = (P_1, P_2, P_3) \in \tilde{\mathscr{R}}$, then

$$\pi^{-1}(P_1, P_2, P_3) = \{(Q_1, Q_2), (-Q_1, Q_2), (Q_1, -Q_2), (-Q_1, -Q_2)\}. \tag{5.5}$$

Let $\tilde{V}$ denote the set of all triples $(\lambda_1, \lambda_2, \lambda_3)$ of non-zero complex numbers satisfying (5.3) and $\tilde{\Phi} : \mathscr{P} \to \tilde{V}$ denote the mapping defined by $\tilde{\Phi}(Q_1, Q_2) = (\operatorname{tr} Q_1, \operatorname{tr} Q_2, \operatorname{tr} Q_1 Q_2)$.

We like to employ $(\lambda_1, \lambda_2, \lambda_3) = \tilde{\Phi}(Q_1, Q_2)$ as the coordinates for $[\rho] = [(P_1, P_2, P_3)] \in \mathscr{R}$ when $\pi(Q_1, Q_2) = (P_1, P_2, P_3)$. However, due to (5.5) $\tilde{\Phi}$ does not induce a well-defined mapping from $\tilde{\mathscr{R}}$ to $\tilde{V}$. So we introduce the equivalence relation on $\tilde{V}$ by $(\lambda_1, \lambda_2, \lambda_3) \sim (\lambda_1', \lambda_2', \lambda_3')$ if and only if $(\lambda_1', \lambda_2', \lambda_3')$ is identical with one of the following points.

$$(\lambda_1, \lambda_2, \lambda_3), \quad (\lambda_1, -\lambda_2, -\lambda_3), \quad (-\lambda_1, \lambda_2, -\lambda_3), \quad (-\lambda_1, -\lambda_2, \lambda_3)$$

Then, $\tilde{\Phi}$ induces a mapping $\Phi$ from $\mathscr{R}$ to the set $V$ of all equivalence classes.

The mapping $\Phi : \mathscr{R} \to V$ gives a global coordinate-system of $\mathscr{R}$, for each point $[(\lambda_1, \lambda_2, \lambda_3)]$ restores the conjugacy class of $(P_1, P_2, P_3)$ uniquely. First it is known that $\lambda_1 = \operatorname{tr} Q_1$ and $\lambda_2 = \operatorname{tr} Q_2$ and $\lambda_3 = \operatorname{tr} Q_1 Q_2$ determine uniquely the conjugacy class of $(Q_1, Q_2)$ satisfying (5.1) (See, e.g. [1, Exercise 4.6]). Then by using (5.4) we can find $P_1$, $P_2$, $P_3$ uniquely under some normalization condition. Actually, under the condition that

$$(P_1 P_2 P_3)^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

and $P_1$ fixes 0, we find

$$P_1 = \begin{pmatrix} -1 & 0 \\ -\lambda_2^2 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 + \lambda_1^2 - \dfrac{\lambda_1 \lambda_3}{\lambda_2} & \dfrac{\lambda_1^2}{\lambda_2^2} \\ -(\lambda_1 \lambda_2 - \lambda_3)^2 & -1 - \lambda_1^2 + \dfrac{\lambda_1 \lambda_3}{\lambda_2} \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -1 + \dfrac{\lambda_1 \lambda_3}{\lambda_2} & \dfrac{\lambda_3^2}{\lambda_2^2} \\ -\lambda_1^2 & -1 - \dfrac{\lambda_1 \lambda_3}{\lambda_2} \end{pmatrix},$$

and in the process of finding them we obtain

$$Q_1 = \begin{pmatrix} \lambda_1 - \dfrac{\lambda_3}{\lambda_2} & \dfrac{\lambda_1}{\lambda_2^2} \\ \lambda_1 & \dfrac{\lambda_3}{\lambda_2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\dfrac{1}{\lambda_2} \\ \lambda_2 & \lambda_2 \end{pmatrix}.$$

Note that $P_1$, $P_2$, $P_3$ depend only on the equivalence class of $(\lambda_1, \lambda_2, \lambda_3)$.

**5.4.** *The mapping class group.* We identify the mapping class group $\mathcal{MC}$ of $F'$ with a subgroup of the outer automorphism group of $G$. Let $\varphi_1, \varphi_2 \in \mathcal{MC}$ be the automorphisms of $G$ which act on $\tilde{\mathcal{R}}$ by

$$\varphi_1 : (P_1, P_2, P_3) \mapsto (P_1, P_2P_3P_2^{-1}, P_2),$$
$$\varphi_2 : (P_1, P_2, P_3) \mapsto (P_3, P_1, P_2).$$

In order to describe the transformations on $V$ induced by $\varphi_i$ for each $i = 1, 2$, we let $(P_1', P_2', P_3') = \varphi_i(P_1, P_2, P_3)$ and find matrices $Q_1'$, $Q_2'$ satisfying

$$P_1'P_2' = -Q_1'^2, \quad P_2'P_3' = -Q_2'^2, \quad P_3'P_1' = -(Q_1'^{-1}Q_2'^{-1})^2$$

and such that $(Q_1', Q_2', Q_1'^{-1}Q_2'^{-1})$ is a $(-)$-system. Let again $Q_5 = Q_1^{-1}Q_2^{-1}$ and $Q_6 = Q_2^{-1}Q_1$. For $\varphi_1$, note that

$$P_1(P_2P_3P_2^{-1}) = (P_2P_4)^{-1} = -(P_2Q_6^{-1}P_2^{-1})^2, \quad (P_2P_3P_2^{-1})P_2 = -Q_2^2$$
$$P_2P_1 = -(P_2Q_1P_2^{-1})^2.$$

Since $Q_2P_2^{-1}$ is elliptic of order 2, (5.2) yields

$$\begin{aligned}\mathrm{tr}(P_2Q_6^{-1}P_2^{-1})Q_2(P_2Q_1P_2^{-1}) &= \mathrm{tr}\, Q_2P_2^{-1}Q_2P_2\\ &= -\mathrm{tr}\, P_2^2 = -2.\end{aligned}$$

Thus $(P_2Q_6^{-1}P_2^{-1}, Q_2, P_2Q_1P_2^{-1})$ is $(-)$-system. So we can let $Q_1' = P_2Q_6^{-1}P_2^{-1}$ and $Q_2' = Q_2$. For $\varphi_2$, it holds that

$$P_3P_1 = -Q_5^2, \quad P_1P_2 = -Q_1^2, \quad P_2P_3 = -Q_2^2.$$

Since $(Q_5, Q_1, Q_2)$ is a $(-)$-system, we can let $Q_1' = Q_5$ and $Q_2' = Q_1$. We define $\tilde{\varphi}_i : \mathcal{P} \to \mathcal{P}$ for $i = 1, 2$ by

$$\tilde{\varphi}_1 : (Q_1, Q_2) \mapsto (P_2Q_6^{-1}P_2^{-1}, Q_2),$$
$$\tilde{\varphi}_2 : (Q_1, Q_2) \mapsto (Q_5, Q_1).$$

Then $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ induce the following transformations on $\tilde{V}$:

$$\begin{aligned}\varphi_{1*}(\lambda_1, \lambda_2, \lambda_3) &= \tilde{\Phi}(\tilde{\varphi}_1(Q_1, Q_2)) = (\lambda_1\lambda_2 - \lambda_3, \lambda_2, \lambda_1),\\ \varphi_{2*}(\lambda_1, \lambda_2, \lambda_3) &= \tilde{\Phi}(\tilde{\varphi}_2(Q_1, Q_2)) = (\lambda_3, \lambda_1, \lambda_2)\end{aligned} \tag{5.6}$$

Here we used Theorem 0.1 to deduce $\mathrm{tr}\, P_2Q_6^{-1}P_2^{-1} = \mathrm{tr}\, Q_6 = (\lambda_1^2 + \lambda_2^2)/\lambda_3$, which equals $\lambda_1\lambda_2 - \lambda_3$ by (5.3). (We can deduce (5.6) directly from (5.3) as well.) Note that $\varphi_{1*}$ and $\varphi_{2*}$ preserve equivalence classes $[\lambda_1, \lambda_2, \lambda_3]$.

We consider the mapping class $\varphi = (\varphi_2^{-1}\varphi_1\varphi_2) \circ (\varphi_2^{-2}\varphi_1\varphi_2^2) \circ \varphi_1^5$. Let $(\Lambda_1, \Lambda_2, \Lambda_3) = \varphi_*(\lambda_1, \lambda_2, \lambda_3)$. Then $\Lambda_1 = \lambda_2$. We solve the equations

$$(\Lambda_1 - \lambda_1, \Lambda_2 - \lambda_2, \Lambda_3 - \lambda_3) = (0, 0, 0)$$

to find the fixed points of $\varphi_*$. Since $\Lambda_1 = \lambda_2$, we let $\lambda_1 = \lambda_2$. Then we find that the last two entries of $\varphi_*(\lambda_1, \lambda_1, \lambda_3) - (\lambda_1, \lambda_1, \lambda_3)$ have the unique common factor

$$\lambda_1^4 - 3\lambda_1^2 - \lambda_1^2\lambda_3 + 2\lambda_3.$$

By solving $\lambda_1^4 - 3\lambda_1^2 - \lambda_1^2\lambda_3 + 2\lambda_3 = 0$ and (5.3) we see that $\varphi_*$ fixes the points

$$\tau = (\sqrt{z}, \sqrt{z}, 2z-4), \quad \text{where } z = (5+\sqrt{7}i)/2,$$

$(-\sqrt{z}, -\sqrt{z}, 2z-4)$ and their complex conjugates. The fixed point $\tau$ corresponds to $(A,B,C) \in \tilde{\mathscr{R}}$ defined by

$$A = \begin{pmatrix} -1 & 0 \\ -z & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3-z & 1 \\ -8+3z & -5+z \end{pmatrix}, \quad C = \begin{pmatrix} -5+2z & -6+2z \\ -z & 3-2z \end{pmatrix}.$$

Since $z^2 = 5z-8$, the group generated by $A$, $B$ and $C$ is a subgroup of $SL(2,\mathbf{Z}[z])$ and hence discrete. Since $\varphi_*$ fixes $\tau$, $(\varphi(A), \varphi(B), \varphi(C))$ and $(A,B,C)$ differ only by a simultaneous conjugation by an element of $SL(2,\mathbf{C})$. Actually $\varphi(A)$, $\varphi(B)$, $\varphi(C)$ are sent to

$$P = BCBCB^{-1}C^{-1}B^{-1}, \quad Q = BCBC^{-1}B^{-1}, \quad R = C^{-1}B^{-1}ABC,$$

respectively, by an inner automorphim of $G$. Let

$$T = \begin{pmatrix} 1 & (z-3)/2 \\ 0 & 1 \end{pmatrix}.$$

Then we have $P = TAT^{-1}$, $Q = TBT^{-1}$ and $R = TCT^{-1}$. The group $\Gamma$ generated by $A$, $B$, $C$ and $T$ acts on the upper-half space model of the hyperbolic 3-space $\mathbf{H}^3$ and $\mathbf{H}^3/\Gamma$ fibers over the circle with fibre the four-times punctured 2-sphere. Let

$$D = (ABC)^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then we have $DAD^{-1} = TCT^{-1}$ and $DT = TD$. Hence $C = B^{-1}A^{-1}D^{-1} = T^{-1}DAD^{-1}T$ and $B = A^{-1}T^{-1}A^{-1}D^{-1}T$. From $Q = TBT^{-1}$ we see that $B$ and $DAT$ are commutative. Hence $\Gamma$ has the following presentation:

$$\Gamma = \langle A, D, T : DTD^{-1}T^{-1} = DATA^{-1}T^{-1}A^{-1}D^{-1}A^{-1}T^{-1}ATA = 1\rangle.$$

Let $L$ be the link in the 3-sphere $S^3$ as depicted in Figure 1. The arrows $a$, $b$ and $c$ are the Wirtinger generators. Then the link group $\pi_1(S^3 - L)$ has the Wirtinger presentation:

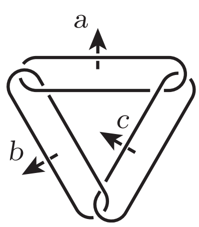


FIGURE 1

$$\langle a, b, c : bab^{-1} = c^{-1}ac, ca^{-1}cab^{-1}a^{-1}c^{-1}aba^{-1}c^{-1}a = 1 \rangle.$$

The mapping sending $A$, $T$ and $D$ to $c$, $a^{-1}$ and $ab^{-1}c^{-1}$, respectively, defines a group isomorphism from $\Gamma$ to $\pi_1(S^3 - L)$. Therefore, by Mostow's rigidity theorem $\mathbf{H}^3/\Gamma$ is homeomorphic to $S^3 - L$. See also [1, Section 9.2].

## References

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