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A Generalization of Wiener's Lemma and its Application to Volterra Difference Equations on a Banach Space

TETSUO FURUMOCHI[a], SATORU MURAKAMI[b,*] and YUTAKA NAGABUCHI[c]

[a]*Department of Mathematics, Shimane University, Matsue 690-8504, Japan;* [b]*Department of Applied Mathematics, Okayama University of Science, 1-1 Ridaicho, Okayama 700-0005, Japan;* [c]*Department of Systems and Control Engineering, Anan National College of Technology, Anan Tokushima 774-0017, Japan*

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Dedicated to Professor Saber Elaydi on the Occasion of His 60th Birthday

Some stability properties for the zero solution of linear Volterra difference equations on a Banach space are studied in connection with the summability of the fundamental solution and the invertibility of the characteristic operator. A key is to extend Wiener's lemma for absolutely convergent scalar sequences to summable sequences of bounded linear operators on a Banach space by applying certain result in commutative Banach algebras.

Keywords: Volterra difference equations; Wiener's lemma; Stabilities; Piecewise continuous delays; Solution operator; Z-transform

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INTRODUCTION

Let X be a Banach space over the field $\mathbf{C}$ of all complex numbers and consider the Volterra difference equation on X

$$x(n+1) = \sum_{j=-\infty}^{n} Q(n-j)\,x(j), \quad n \in \mathbf{Z}^+ := \{0, 1, 2, \ldots\}, \tag{1}$$

where $Q(n)$, $n \in \mathbf{Z}^+$, are bounded linear operators on X such that $\sum_{n=0}^{\infty} \|Q(n)\| < \infty$.

Equation (1) is derived in a natural manner from certain abstract differential equations with piecewise continuous delays as we show in section "Examples and Some Remarks" (see also Refs. [6–8,11]). So the study for the asymptotic behavior of solutions of such differential equations is reduced to that for those of the Volterra difference equation (1).

When X is of finite dimension, stability properties for Eq. (1) have been discussed in Refs. [3,4], and characterized in connection with the invertibility of the characteristic operator, $zI - \sum_{n=0}^{\infty} Q(n) z^{-n}$, assosiated with Eq. (1).

In this paper, we study the uniform asymptotic stability and the exponential stability of the zero solution of Eq. (1), and extend specifically the results in Refs. [3,4] for the case of finite

*Corresponding author. E-mail: murakami@youhei.xmath.ous.ac.jp

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dimensional X to the case of infinite dimensional X. We note that in Ref. [4] Wiener's lemma on the Banach algebra, the ℓ^1-space of sequences of matrices, played quite an important role in order to establish the uniform asymptotic stability for Eq. (1).

In the second section, we generalize Wiener's lemma to the Banach algebra which consists of sequences of bounded linear operators on infinite dimensional X. Applying this infinite-dimensional version of Wiener's lemma, we prove the summability of the fundamental solution of Eq. (1) under some conditions on the characteristic operator. In third section we give necessary and sufficient conditions for the zero solution to be uniformly asymptotically stable. In fourth section we discuss the exponential stability of the zero solution and show that the uniform asymptotic stability need not imply the exponential stability for the Volterra difference equation. More precisely, we prove that if the zero solution is uniformly asymptotically stable, it is exponentially stable if and only if the coefficients $\{Q(n)\}$ decays exponentially. Finally, analyzing the spectrum of the characteristic operator, we show applications of our results to some special differential equations with piecewise continuous delays.

A GENERALIZATION OF WIENER'S LEMMA AND SUMMABILITY OF THE FUNDAMENTAL SOLUTION

For a Banach space X (with norm $|\cdot|$) over the field $\mathbf{C}$, we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X, and define the norm of any T belonging to $\mathcal{L}(X)$ by

$$\|T\| = \sup\{|Tx| \, : \, x \in X, \ |x| = 1\}.$$

Let $L^1(\mathbf{Z}^+)$ be the space of all sequences $Q := \{Q(n)\} = (Q(0), Q(1), Q(2), \ldots)$ with $Q(j) \in \mathcal{L}(X)$, $j \in \mathbf{Z}^+$, satisfying

$$\sum_{n=0}^{\infty} \|Q(n)\| < \infty.$$

For any Q and W in $L^1(\mathbf{Z}^+)$, we define the product $Q*W$ by

$$(Q*W)(n) = \sum_{k=0}^{n} Q(n-k)W(k), \quad n \in \mathbf{Z}^+.$$

One can easily see that the space $L^1(\mathbf{Z}^+)$ with the product defined above is a (non-commutative) Banach algebra equipped with norm

$$\|Q\| = \sum_{n=0}^{\infty} \|Q(n)\|.$$

In fact, $L^1(\mathbf{Z}^+)$ possesses the element $e_0 =: e$ defined by

$$e_0(0) = I, \quad e_0(n) = 0 \quad (n = 1, 2, \ldots)$$

as the unit, where I denotes the identity operator on X.

For each $j \in \mathbf{Z}^+$, let us consider an element e_j defined by the relation

$$e_j(j) = I, \quad e_j(n) = 0 \quad (n \neq j).$$

Then it follows that $e_j * e_i = e_{j+i}$, $(\forall i, j \in \mathbf{Z}^+)$. Moreover, any element $a = \{a_n\}$ in $L^1(\mathbf{Z}^+)$ is expressed as

$$a = \sum_{n=0}^{\infty} a_n e_n = \sum_{n=0}^{\infty} a_n E^n,$$

where $E := e_1$ and $E^n := E * E * \cdots * E$ (the product of n copies of E).

Wiener's lemma is generalized as follows.

THEOREM 1 *Assume that* $Q = \{Q(n)\} \in L^1(\mathbf{Z}^+)$ *satisfies the following two conditions:*

(i) $Q(i)Q(j) = Q(j)Q(i)$ *for* $i, j \in \mathbf{Z}^+$;
(ii) for any $|z| \leq 1$, *the operator* $\sum_{k=0}^{\infty} Q(k)z^k$ *is invertible in* $\mathcal{L}(X)$.

Then Q *is invertible in* $L^1(\mathbf{Z}^+)$; *in other words, there exists an* $R = \{R(n)\} \in L^1(\mathbf{Z}^+)$ *such that*

$$Q*R = R*Q = e_0.$$

Proof *Fix any* z *such that* $|z| \leq 1$. *By the condition* (ii), *the operator* $T := \sum_{k=0}^{\infty} Q(k)z^k$ *has a bounded inverse* T^{-1} *in* $\mathcal{L}(X)$. *It follows that*

$$(T, 0, 0, \ldots)*(T^{-1}, 0, 0, \ldots) = (T^{-1}, 0, 0, \ldots)*(T, 0, 0, \ldots) = e_0,$$

and hence $(T,0,0,\ldots)$ *is invertible in* $L^1(\mathbf{Z}^+)$.

Now, let us consider the subset Ω *of* $L^1(\mathbf{Z}^+)$ *which consists of all the elements of the form* $(0,\ldots,0,Q(\cdot),0,0,\ldots)$, *and set*

$$Y = \Gamma(\Gamma(\Omega)),$$

where $\Gamma(\mathcal{C})$ denotes the centralizer of the set $\mathcal{C}$, that is

$$\Gamma(\mathcal{C}) = \left\{ W \in L^1(\mathbf{Z}^+) \ : \ W*P = P*W \quad \text{for any} \ \ P \in \mathcal{C} \right\}.$$

Since the set Ω *commutes by the condition (i), it follows from Ref.* [10, *p.* 280, *Theorem* 11.22] *that* $Y = \Gamma(\Gamma(\Omega))$ *is a commutative Banach subalgebra containing* Ω. *Let* χ *be any character of* Y, *and set* $z_0 = \chi(e_1)$. *Then* $z_0 \in \mathbf{C}$ *with* $|z_0| \leq 1$. *By virtue of the condition (ii), the element* $\sum_{k=0}^{\infty} Q(k)z_0^k$ *is invertible in* $\mathcal{L}(X)$, *and hence* $(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \ldots)$ *is invertible in* $L^1(\mathbf{Z}^+)$ *by the fact mentioned in the first paragraph of this proof; consequently, from Ref.* [10, *p.* 280, *Theorem* 11.22] *it follows that* $(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \ldots)$ *is invertible in* Y. *In particular, we get*

$$\chi\left(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \ldots\right) \neq 0.$$

Hence

$$\begin{aligned}
\chi(Q) &= \chi\left(Q(0), Q(1), Q(2), \ldots\right) \\
&= \chi((Q(0), 0, 0, \ldots)*e_0 + (Q(1), 0, 0, \ldots)*e_1 + (Q(2), 0, 0, \ldots)*e_1*e_1 + \cdots) \\
&= \sum_{k=0}^{\infty} \chi((Q(k), 0, 0, \ldots))\{\chi(e_1)\}^k \\
&= \chi\left(\sum_{k=0}^{\infty} Q(k)z_0^k, 0, 0, \ldots\right) \neq 0,
\end{aligned}$$

which shows that $Q = \{Q(n)\}$ *does not belong to any maximal ideal of* Y. *Then Ref.* [10, *p.* 265, *Theorem* 11.5] *yields that* Q *is invertible in* Y, *and so is it in* $L^1(\mathbf{Z}^+)$. *This completes the proof of the theorem.* □

We now consider the Volterra difference equation (1) on X with $Q(n) \in \mathcal{L}(X)$, $n \in \mathbf{Z}^+$. We assume the following condition on Eq. (1):

$$\exists\, l \geq 1 : \quad \sum_{n=0}^{\infty} \|Q(n)\| \leq l < \infty.$$

For Eq. (1) we define $R(n)$, $n \in \mathbf{Z}^+$, in $\mathcal{L}(X)$ by the relation

$$R(n+1) = \sum_{j=0}^{n} Q(n-j)\,R(j), \quad n = 0, 1, 2, \ldots, \tag{2}$$

and $R(0) = I$. The sequence $R(n)$, $n \in \mathbf{Z}^+$, is called *the fundamental solution* of Eq. (1). By the virtue of the inequality

$$\|R(n+1)\| = \|Q(n)\,R(0) + Q(n-1)\,R(1) + \cdots + Q(0)\,R(n)\| \leq l \max_{0 \leq s \leq n} \|R(s)\|,$$

we get the following estimate on $\|R(n)\|$:

$$\|R(n)\| \leq l^n, \quad n \in \mathbf{Z}^+.$$

Therefore, the Z-transform

$$\tilde{\mathrm{R}}(z) = \sum_{n=0}^{\infty} R(n) z^{-n}$$

of $R(n)$, $n \in \mathbf{Z}^+$, exists in the domain $|z| > l$ of the complex plane $\mathbf{C}$, and it is analytic in the domain.

We say that $R(n)$, $n \in \mathbf{Z}^+$, is summable, if $R = \{R(n)\} \in L^1(\mathbf{Z}^+)$, that is, $\sum_{n=0}^{\infty} \|R(n)\| < \infty$. For the summability of the fundamental solution of Eq. (1) we get:

COROLLARY 1 *Assume that the coefficients $Q = \{Q(n)\} \in L^1(\mathbf{Z}^+)$ in Eq.* (1) *satisfies the condition* (*i*) *in Theorem* 1, *together with the following condition:*

(*ii′*) *for any $|z| \geq 1$, the characteristic operator of Eq.* (1) $zI - \sum_{n=0}^{\infty} Q(n) z^{-n}$ *is invertible in $\mathcal{L}(X)$.*

Then the fundamental solution $\{R(n)\}$ of Eq. (1) *is summable.*

Proof *By considering the Z-transform of both hand sides of Eq.* (2), *we get*

$$z\tilde{R}(z) - zI = \tilde{Q}(z)\,\tilde{R}(z), \quad |z| > l,$$

or

$$\left(I - \frac{1}{z}\tilde{Q}(z)\right)\tilde{R}(z) = I, \quad |z| > l.$$

Hence it follows from the condition (ii′) *that*

$$\left(I - \frac{1}{z}\tilde{Q}(z)\right)^{-1} = \tilde{R}(z), \quad |z| > l.$$

Now we define $S = \{S(n)\}$ by the relation

$$S(0) = I, \quad S(n) = -Q(n-1) \quad n = 1, 2, \ldots.$$

Observe that $\tilde{S}(z) = \left(I - \frac{1}{z}\tilde{Q}(z)\right)$ *for* $|z| \geq 1$*, and consequently, the conditions* (i) *and* (ii) *in Theorem* 1 *are satisfied for* $S = \{S(n)\}$*. Thus there exists an* $H = \{H(n)\} \in L^1(\mathbf{Z}^+)$ *such that* $H{*}S = S{*}H = e_0$*, and hence*

$$\tilde{H}(z)\tilde{S}(z) = \tilde{S}(z)\tilde{H}(z) = I, \quad |z| \geq 1.$$

Consequently, we get

$$\tilde{R}(z) = \tilde{H}(z), \quad |z| > l,$$

which implies that $R(n) \equiv H(n)$ *by the uniqueness of the Laurent expansion. Hence* $R = \{R(n)\}$ *belongs to* $L^1(\mathbf{Z}^+)$*, or it is summable. This completes the proof of the corollary.* □

UNIFORM ASYMPTOTIC STABILITY

Let $\mathbf{Z}^-$ be the set of all nonpositive integers and consider the Banach space $\mathcal{B}$ defined by

$$\mathcal{B} = \left\{ \phi : \mathbf{Z}^- \mapsto X \,\middle|\, \sup_{\theta \in \mathbf{Z}^-} |\phi(\theta)| < \infty \right\}$$

equipped with the norm

$$\|\phi\| = \sup_{\theta \in \mathbf{Z}^-} |\phi(\theta)|, \quad \phi \in \mathcal{B}.$$

Since

$$\sum_{j=-\infty}^{-1} \|Q(n-j)\phi(j)\| \leq \left(\sum_{j=n+1}^{\infty} \|Q(j)\| \right) \|\phi\| \leq \|Q\| \, \|\phi\| \quad \text{for } \phi \in \mathcal{B},$$

we see that

$$p(n) := \sum_{j=-\infty}^{-1} Q(n-j)\phi(j), \quad n \in \mathbf{Z}^+$$

is well-defined for $\phi \in \mathcal{B}$ and that for any $\tau \in \mathbf{Z}^+$, Eq. (1) has a unique solution $x(n)$ for $n \geq \tau$ satisfying the initial condition $x(\tau + \theta) \equiv \phi(\theta)$, $\theta \in \mathbf{Z}^-$. We denote this solution by $x(n;\tau,\phi)$. Since

$$\begin{aligned} x(n+\tau+1;\tau,\phi) &= \sum_{j=-\infty}^{n+\tau} Q(n+\tau-j)\, x(j;\tau,\phi) \\ &= \sum_{j=0}^{n} Q(n-j)\, x(j+\tau,\phi) + \sum_{j=-\infty}^{\tau-1} Q(n+\tau-j)\, \phi(j-\tau), \end{aligned}$$

$y(n) := x(n+\tau;\tau,\phi)$ satisfies the equation

$$y(n+1) = \sum_{j=0}^{n} Q(n-j)y(j) + p(n), \quad n \in \mathbf{Z}^+.$$

Then the variation-of-constants formula gives

$$y(n) = R(n)y(0) + \sum_{j=0}^{n-1} R(n-j-1)\, p(j), \quad n \in \mathbf{Z}^+.$$

Thus we get

$$x(n;\tau,\phi) = R(n-\tau)\phi(0) + \sum_{j=\tau}^{n-1} R(n-j-1)\left(\sum_{s=-\infty}^{-1} Q(j-\tau-s)\phi(s)\right) \tag{3}$$

for $n \geq \tau$, where we promise that $\sum_{j=\tau}^{\tau-1} = 0$ for $\tau \geq 0$.

For stabilities of the zero solution we follow the standard definitions below:

DEFINITION 1 *(i) The zero solution of Eq.* (1) *is said to be uniformly asymptotically stable if the following two properties are satisfied:*

(Uniform stability) For any $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that if* $\tau \geq 0$ *and* $\phi \in \mathcal{B}$ *satisfies* $\|\phi\| < \delta$, *then* $|x(n;\tau,\phi)| < \varepsilon$ *for* $n \geq \tau$.

(Uniform attractivity) There exists a $\kappa > 0$ *such that for any* $\varepsilon > 0$ *there exists an* $N = N(\varepsilon) \in \mathbf{Z}^+$ *such that if* $\tau \geq 0$ *and* $\phi \in \mathcal{B}$ *satisfies* $\|\phi\| < \kappa$, *then* $|x(n;\tau,\phi)| < \varepsilon$ *for* $n \geq \tau + N$.

(ii) The zero solution of Eq. (1) *is said to be (globally) exponentially stable if there exist constants* $M > 0$ *and* $\nu \in (0,1)$ *such that*

$$|x(n;\tau,\phi)| \leq M\nu^{n-\tau}\|\phi\| \quad \text{for } n \geq \tau \quad \text{and } \phi \in \mathcal{B}.$$

We are now in a position to state our results for the uniform asymptotic stability of Eq. (1).

THEOREM 2 *Assume that the coefficients* $Q = \{Q(n)\} \in L^1(\mathbf{Z}^+)$ *in Eq.* (1) *satisfy the condition* (*i*) *in Theorem* 1 *and that* $Q(n)$, $n \in \mathbf{Z}^+$, *are all compact. Then, for Eq.* (1) *the following statements are equivalent.*

(i) $(zI - \tilde{Q}(z))^{-1} \in \mathcal{L}(X)$ for $|z| \geq 1$.

(ii) $\{R(n)\} \in L^1(\mathbf{Z}^+)$.

(iii) The zero solution of Eq. (1) *is uniformly asymptotically stable.*

Proof (*i*) $\Rightarrow$ (*ii*). *This is an immediate consequence of Corollary* 1.

(*ii*) $\Rightarrow$ (*iii*). *Assume that* $\{R(n)\} \in L^1(\mathbf{Z}^+)$. *It follows from Eq.* (3) *that*

$$\begin{aligned} |x(n+\tau;\tau,\phi)| &= \left| R(n)\phi(0) + \sum_{j=\tau}^{n+\tau-1} R(n+\tau-j-1)\left(\sum_{s=-\infty}^{-1} Q(j-\tau-s)\phi(s)\right)\right| \\ &\leq \|\phi\|\left[\|R(n)\| + \sum_{j=0}^{n-1}\|R(n-j-1)\|\sum_{s=-\infty}^{-1}\|Q(j-s)\|\right] \end{aligned}$$

for any $n \geq 0$, $\tau \geq 0$ *and* $\phi \in \mathcal{B}$. *The term* $\|R(n)\| + \sum_{j=0}^{n-1}\|R(n-j-1)\|\sum_{s=-\infty}^{-1}\|Q(j-s)\|$ *is dominated by* $\|R\|(1+\|Q\|)$. *Moreover, the term* $\sum_{j=0}^{n-1}\|R(n-j-1)\|\sum_{s=-\infty}^{-1}\|Q(j-s)\|$ *tends to* 0 *as* $n \to \infty$, *because it is the convolution of an* ℓ^1*-function with one which tends to* 0 *as* $n \to \infty$. *These observations lead to the uniform asymptotic stability of the zero solution of Eq.* (1).

(*iii*) $\Rightarrow$ (*i*). *Assume that the zero solution of Eq.* (1) *is uniformly asymptotically stable, and that (i) is not true. Then there exists some* z_0 *with* $|z_0| \geq 1$ *such that* $(z_0 I - \tilde{Q}(z_0))^{-1} \notin \mathcal{L}(X)$. *We note that* $\tilde{Q}(z_0) = \sum_{j=0}^{\infty} Q(j)z_0^{-j}$ *is a compact operator, because it is the limit of a sequence of compact operators* $\{\sum_{j=0}^{n} Q(j)z_0^{-j}\}$ *in the operator norm as* $n \to \infty$.

It follows from the Riesz-Schauder theorem on compact operators that z_0 is an eigenvalue of $\tilde{Q}(z_0)$. Hence there exists a nonzero element $x_0 \in X$ with $|x_0| = 1$ such that $(z_0 I - \tilde{Q}(z_0))x_0 = 0$, or equivalently

$$z_0 x_0 = \sum_{n=0}^{\infty} Q(n) x_0 z_0^{-n}. \tag{4}$$

Put $x(n) := (\kappa/2) z_0^n x_0$, where κ is the constant in the uniform attractivity of the zero solution of Eq. (1). *Using Eq.* (4) *we get*

$$x(n+1) = (\kappa/2) z_0^{n+1} x_0 = (\kappa/2) z_0^n \sum_{j=0}^{\infty} Q(j) x_0 z_0^{-j} = \sum_{j=0}^{\infty} Q(j) x(n-j),$$

and hence $x(n)$ is a solution of Eq. (1). *Notice that $|x(n)| \le \kappa/2 < \kappa$ for $n \in \mathbf{Z}^-$. Since the zero solution of Eq. (1) is uniformly asymptotically stable, we get that $|x(N)| < \kappa/2$, where $N := N(\kappa/2)$. This is a contradiction, because $|x(n)| \ge \kappa/2$ for $n \in \mathbf{Z}^+$. The proof is now complete.* □

Remark 1 The implication (i) $\Rightarrow$ (ii) in Theorem 2 holds true under a weaker assumption. Indeed, following the proof of the part in Theorem 2 one can see that the implication holds true without the compactness condition on $Q(n)$, $n \in \mathbf{Z}^+$. Also, the implication (iii) $\Rightarrow$ (i) holds true without the condition (i) in Theorem 1.

EXPONENTIAL STABILITY

In this section we discuss the exponential stability of the zero solution of Eq. (1) to get an extension of Ref. [4, Theorems 4 and 5] to the case of infinite dimensional X.

An element $\{R(n)\} \in L^1(\mathbf{Z}^+)$ is said to *decay exponentially*, if there exist positive constants M and ν with $0 < \nu < 1$ such that $\|R(n)\| \le M\nu^n$ for $n \in \mathbf{Z}^+$.

THEOREM 3 *Let $Q(n)$, $n \in \mathbf{Z}^+$, be compact operators, and assume that $\|R(n)\|$ tends to zero as $n \to \infty$. Then $R(n)$ decays exponentially if and only if so does $Q(n)$.*

Proof *Suppose that $\|R(n)\| \le M\nu^n$ holds for $n \in \mathbf{Z}^+$ with some constants $M > 0$ and $\nu \in (0,1)$. Then $\tilde{R}(z) = \sum_{n=0}^{\infty} R(n) z^{-n}$ is absolutely convergent for $|z| > \nu$. Let us consider the Z-transform of Eq.* (2) *to obtain*

$$(zI - \tilde{Q}(z))\,\tilde{R}(z) = zI \quad \text{for } |z| \ge 1.$$

It follows that $zI - \tilde{Q}(z)$ is surjective. Recall that $\tilde{Q}(z)$ is compact. Then the Riesz–Schauder theory implies that $zI - \tilde{Q}(z)$ is also injective. Therefore, $\tilde{R}(z)$ is invertible in $\mathcal{L}(X)$ for $|z| \ge 1$, and consequently, $\tilde{R}(z)$ has its inverse in $\mathcal{L}(X)$ for each z in some open neighborhood of the set $|z| = 1$. Hence there is a positive constant δ with $\delta < 1 - \nu$ such that $\tilde{R}(z)$ is invertible in $\mathcal{L}(X)$ for any z with $|z| \ge 1 - \delta$. Since $\tilde{R}(z)$ is analytic on the domain $|z| > 1 - \delta$, so is $\tilde{R}(z)^{-1}$. Let us consider an analytic function $F(z)$ defined by $F(z) = zI - z\tilde{R}(z)^{-1}$ on the domain $|z| > 1 - \delta$, and denote the Laurent expansion of $F(z)$ by

$$F(z) = \sum_{n \in \mathbf{Z}} b(n) z^n, \quad |z| > 1 - \delta,$$

where

$$b(n) = \frac{1}{2\pi i}\int_{|z|=L} \frac{F(z)}{z^{n+1}}\mathrm{d}z, \quad L > 1-\delta.$$

Since $F(z) = \tilde{Q}(z)$ *for* $|z| \geq 1$, *it follows that*

$$\sup_{|z|\geq 1} \|F(z)\| = \sup_{|z|\geq 1}\left|\sum_{n=0}^{\infty} Q(n)z^{-n}\right| \leq \sum_{n=0}^{\infty}\|Q(n)\| = \|Q\|.$$

Therefore, we have

$$\|b(n)\| = \left\|\frac{1}{2\pi i}\int_{|z|=L}\frac{F(z)}{z^{n+1}}\mathrm{d}z\right\| \leq \frac{1}{2\pi}\frac{1}{L^{n+1}}\left(\sup_{|z|=L}\|F(z)\|\right)\times 2\pi L \leq \frac{\|Q\|}{L^n}$$

for $L \geq 1$. *Letting* $L \to \infty$, *we get that* $b(n) = 0$ $(n = 1, 2, \ldots)$, *and hence*

$$F(z) = \sum_{n=0}^{\infty} b(-n)z^{-n}, \quad |z| > 1-\delta.$$

In particular, the series $\sum_{n=0}^{\infty} b(-n)(1-\delta/2)^{-n}$ *is convergent. Hence we have*

$$\|b(-n)\| \leq M_1\left(1-\frac{\delta}{2}\right)^n, \quad n \in \mathbf{Z}^+ \tag{5}$$

for some constant $M_1 > 0$. *Since*

$$\sum_{n=0}^{\infty} Q(n)z^{-n} = \tilde{Q}(z) = F(z) = \sum_{n=0}^{\infty} b(-n)z^{-n} \quad \text{for} \quad |z| \geq 1,$$

the uniqueness of the Laurent expansion yields that $Q(n) = b(-n)$, $n \in \mathbf{Z}^+$. *This, together with Eq. (5), implies*

$$\|Q(n)\| \leq M_1\left(1-\frac{\delta}{2}\right)^n, \quad n \in \mathbf{Z}^+,$$

which shows that Q(n) decays exponentially.

Conversely, suppose that $\|Q(n)\| \leq M_2\nu_1^n$ *for* $n \in \mathbf{Z}^+$ *with some constants* $M_2 > 0$ *and* $\nu_1 \in (0, 1)$. *It follows from Eq.* (3) *that*

$$x(n+\tau;\tau,\phi) = R(n)\phi(0) + \sum_{j=0}^{n-1} R(n-j-1)\sum_{s=-\infty}^{-1} Q(j-s)\phi(s),$$

and hence

$$\begin{aligned}|x(n+\tau;\tau,\phi) - R(n)\phi(0)| &\leq M_2\|\phi\|\sum_{j=0}^{n-1}\|R(n-j-1)\|\sum_{s=-\infty}^{-1}\nu_1^{j-s}\\ &= \frac{M_2\|\phi\|}{1-\nu_1}\sum_{j=0}^{n-1}\|R(n-j-1)\|\nu_1^{j+1}\end{aligned}$$

for $n \geq 0$, $\tau \geq 0$ *and* $\phi \in \mathcal{B}$. *Since* $\|R(n)\| \to 0$ *as* $n \to \infty$, *we see by the above inequality that the zero solution of Eq.* (1) *is uniformly asymptotically stable. Hence it follows* *from Theorem* 2 *and Remark* 1 *that* $(zI - \tilde{Q}(z))^{-1} \in \mathcal{L}(X)$ *for* $|z| \geq 1$. *By the same reasoning as in the proof of the former part, we can conclude that* $(zI - \tilde{Q}(z))^{-1} \in \mathcal{L}(X)$ *for* $|z| > 1 - \delta_1$ *with some constant* $\delta_1 \in (0,\, 1 - \nu_1)$ *and is analytic there. Now consider a function* $G(z) := z(zI - \tilde{Q}(z))^{-1}$ *on the domain* $|z| > 1 - \delta_1$, *and let* $G(z) = \sum_{n\in\mathbf{Z}} c(n)z^n$, $|z| > 1 - \delta_1$, *be its Laurent expansion. As seen in the paragraph preceding Corollary* 1, *it follows that* $\sup_{n\in\mathbf{Z}^+} \|R(n)\| l^{-n} = 1$. *In particular,* $\tilde{R}(z) = \sum_{n=0}^{\infty} R(n)z^{-n}$ *absolutely converges on the domain* $|z| > l$, *and it satisfies the relation*

$$(zI - \tilde{Q}(z))\,\tilde{R}(z) = zI, \quad |z| > l.$$

Hence we get that $\tilde{R}(z) = G(z)$ *for* $|z| > l$, *and*

$$\begin{aligned}\sup_{|z|\geq 2l} \|G(z)\| &\leq \sup_{|z|\geq 2l} \sum_{n=0}^{\infty} \|R(n)z^{-n}\| \\ &\leq \sum_{n=0}^{\infty} \|R(n)\|(2l)^{-n} \\ &\leq \sum_{n=0}^{\infty} 2^{-n} = 2.\end{aligned}$$

The same argument as for $F(z)$ *gives* $c(n) = 0$ *for* $n = 1, 2, \ldots$, *and* $\|c(-n)\| \leq M_3(1 - \delta_1/2)^n$, $n \in \mathbf{Z}^+$ *for some* $M_3 > 0$. *Since*

$$\sum_{n=0}^{\infty} R(n)z^{-n} = \tilde{R}(z) = G(z) = \sum_{n=0}^{\infty} c(-n)z^{-n} \quad \text{for } |z| > l,$$

it follows from the uniqueness of the Laurent expansion that $R(n) = c(-n)$ *for* $n \in \mathbf{Z}^+$. *In particular, we obtain* $\|R(n)\| \leq M_3(1 - \delta_1/2)^n$, $n \in \mathbf{Z}^+$, *which proves that* $R(n)$ *decays exponentially.* □

THEOREM 4 *Let* $Q(n)$, $n \in \mathbf{Z}^+$, *be compact operators, and assume that the zero solution of Eq.* (1) *is uniformly asymptotically stable. Then the zero solution of Eq.* (1) *is exponentially stable if and only if* $Q(n)$ *decays exponentially.*

Proof *The "only if" part follows directly from Theorems* 2 *and* 3. *We will prove the "if" part. Suppose that* $\|Q(n)\| \leq M_1\nu_1^n$ *for* $n \in \mathbf{Z}^+$ *with constants* M_1 *and* $\nu_1 \in (0, 1)$. *As an easy consequence of the uniform asymptotic stability of the zero solution of Eq.* (1), *one can see that* $\lim_{n\to\infty} \|R(n)\| = 0$, *and hence it follows from Theorem* 3 *that* $R(n)$ *decays exponentially, that is, there are constants* $M_2 > 0$ *and* $\nu_2 \in (\nu_1,\, 1)$ *such that* $\|R(n)\| \leq M_2\nu_2^n$ *for* $n \in \mathbf{Z}^+$. *Then, by Eq.* (3) *we have*

$$\begin{aligned}|x(n;\tau,\phi)| &\leq M_2\|\phi\|\left(\nu_2^{n-\tau} + \sum_{j=\tau}^{n-1} \nu_2^{n-j-1} \sum_{s=-\infty}^{-1} M_1\nu_1^{j-\tau-s}\right) \\ &\leq M_2\|\phi\|\left(1 + \frac{M_1\nu_1}{(1-\nu_1)(\nu_2-\nu_1)}\right)\nu_2^{n-\tau}.\end{aligned}$$

Thus, the zero solution of Eq. (1) *is exponentially stable.* □

EXAMPLES AND SOME REMARKS

In this section, following the idea in Ref. [11], we explain how Volterra difference equations on a Banach space are canonically derived from some abstract differential equations with piecewise continuous delays, also refer to Ref. [5]. Moreover, we analyze the spectrum of the characteristic operator for the induced Volterra difference equations to obtain a condition under which some results established in the previous section are applicable.

In what follows, we employ the notation $[\cdot]$ to denote the Gaussian symbol, and consider the differential equation

$$\dot{u}(t) = Au(t) + \sum_{k=0}^{\infty} B(k)u([t-k]), \quad t \geq 0 \tag{6}$$

on a Banach space X, which contains piecewise continuous delays $t-[t-k]$, $k = 0, 1, 2, \ldots$. Here and hereafter, we assume that A is the inifinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, of bounded linear operators on X, and $B(k)$, $k = 0, 1, 2, \ldots$, are bounded linear operators on X such that

$$\sum_{k=0}^{\infty} \|B(k)\| < \infty. \tag{7}$$

A function $u : (\mathbf{Z}^- \cup [0,\infty)) \mapsto X$ such that $u_0 \in \mathcal{B}$ (that is, $\sup_{\theta \in \mathbf{Z}^-} |u(\theta)| < \infty$) is called a (mild)solution of Eq. (6) on [0,∞), if u is continuous on [0,∞), and it satisfies the following relation

$$u(t) = T(t-\sigma)u(\sigma) + \int_\sigma^t T(t-s)\Big(\sum\nolimits_{k=0}^{\infty} B(k)u([s-k])\Big)\,\mathrm{d}s, \quad t \geq \sigma \geq 0.$$

In case of $n \leq t < n+1$ for some nonnegative integer n, the above relation yields that

$$\begin{aligned} u(t) &= T(t-n)u(n) + \int_n^t T(t-s)\left(\sum_{k=0}^{\infty} B(k)u([s-k])\right)\mathrm{d}s \\ &= T(t-n)u(n) + \sum_{k=0}^{\infty}\left(\int_n^t T(t-s)B(k)u(n-k)\,\mathrm{d}s\right). \end{aligned}$$

Letting $t \to n+1$ in this equation, we get Volterra difference equation

$$u(n+1) = \sum_{k=0}^{\infty} Q(k)u(n-k), \quad n \in \mathbf{Z}^+, \tag{8}$$

where $Q(k)$, $k \in \mathbf{Z}^+$, are bounded linear operators on X defined by

$$Q(0)x = T(1)x + \int_0^1 T(\tau)B(0)x\,\mathrm{d}\tau, \quad Q(k)x = \int_0^1 T(\tau)B(k)x\,\mathrm{d}\tau, \quad k = 1, 2, \ldots \tag{9}$$

for $x \in X$.

Conversely, if u satisfies Eq. (8) with $\sup_{\theta \in \mathbf{Z}^-} |u(\theta)| < \infty$, then the function u extended to non-integers t by the relation

$$u(t) = T(t-n)u(n) + \sum_{k=0}^{\infty}\left(\int_n^t T(t-s)B(k)u(n-k)\,\mathrm{d}s\right), \quad n < t < n+1, \quad n \in \mathbf{Z}^+,$$

is a (mild) solution of Eq. (6). Thus, some abstract differential equations such as Eq. (6) lead to Volterra difference equations on X. Sometimes, we call Eq. (8) *the induced Volterra difference equation* of Eq. (6).

A strongly continuous semigroup $T(t)$ on X is said to be *compact* whenever $T(t)$ is a compact operator on X for $t > 0$. It is known [9] that if the semigroup $T(t)$ is compact, then $T(t)$ is continuous in $t > 0$ with respect to the operator norm.

It is known ([5, Proposition 1]) that $Q(k)$, $k \in \mathbf{Z}^+$, defined by the relation (9) are compact operators on X whenever $T(t)$ is a compact semigroup on X. Moreover, it follows from Eq. (7) that $Q \in L^1(\mathbf{Z}^+)$.

Under the restricted case where $B(k)$, $k \in \mathbf{Z}^+$ are scalar, that is, $B(k) \equiv b(k)I$, $k \in \mathbf{Z}^+$, for some $b(k) \in \mathbf{C}$, we can determine the spectrum of the characteristic operator $zI - \tilde{Q}(z) := zI - \sum_{k=0}^{\infty} Q(k)z^{-k}$ of Eq. (8).

PROPOSITION 1 *Let $T(t)$ be a compact semigroup on X, and assume that $B(k) \equiv b(k)I$, $k \in \mathbf{Z}^+$, where $b(k)$ is a scalar function satisfying $\sum_{k=0}^{\infty}|b(k)| < \infty$. Then the spectrum of the characteristic operator $zI - \tilde{Q}(z)$ with $|z| \geq 1$ of Eq. (8) is given by*

$$\sigma(zI - \tilde{Q}(z)) = \left(\{z\} \cup \left\{z - e^{\nu} - \tilde{b}(z)\int_0^1 e^{\nu\tau}\,\mathrm{d}\tau \mid \ \nu \in \sigma(A)\right\}\right). \tag{10}$$

Proof *We will give an outline of the proof of the proposition; see Ref.* [5, *Theorem* 3] *for the complete proof.*

By using the continuity of $T(t)$ in $t > 0$ with respect to the operator norm, one can see that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}\{T(1/n)\}^k - \int_0^1 T(\tau)\,\mathrm{d}\tau\right\| = 0. \tag{11}$$

Now, set $S = \{T(t) : 0 \leq t \leq 1\}$. Since S commutes, $\mathcal{A} := \Gamma(\Gamma(S))$ is a commutative Banach algebra containing S, see Ref. [10, *p.* 280, *Theorem* 11.22]. *Here, for any subset Ω of $\mathcal{L}(X)$, $\Gamma(\Omega)$ denotes the centralizer of Ω that is,*

$$\Gamma(\Omega) = \{v \in \mathcal{L}(X) : vw = wv \quad \textit{for every } w \in \Omega\}.$$

Let Δ be the maximal ideal space of $\mathcal{A}$. Let us denote by $\hat{a}$ the Gelfand transform of $a \in \mathcal{A}$. It is known [10, pp. 268–270] that $\hat{a}$ is a function from Δ (which is equipped with the Gelfand topology) into $\mathbf{C}$ with the properties that the range of $\hat{a}$ coincides with the spectrum $\sigma(a)$ of a and that

$$\|\hat{a}\|_\infty \leq \|a\|, \quad a \in \mathcal{A},$$

where $\|\hat{a}\|_\infty$ is the maximum of $|\hat{a}(\xi)|$ on Δ. Moreover, the Gelfand transform is a homomorphism mapping $\mathcal{A}$ into a subspace of *$C(\Delta;\mathbf{C})$, the space of all the complex valued continuous functions on Δ. Let $|z| \geq 1$, and put*

$$a = zI - \tilde{Q}(z) = zI - T(1) - \left(\int_0^1 T(\tau)\,\mathrm{d}\tau\right)\hat{b}(z)$$

and

$$a_n = zI - W^n - \left(\frac{1}{n}\sum_{k=1}^{n} W^k\right)\hat{b}(z)$$

for each $n = 1, 2, \ldots,$ *where* $W := T(1/n)$. *Then* $\{a, a_1, a_2, \ldots\} \subset \mathcal{A}$, *and by Eq.* (11) *we get*

$$\|(\widehat{a_n}) - \hat{a}\|_\infty \leq \|a_n - a\| = \left\| \frac{1}{n} \sum_{k=1}^{n} \{T(1/n)\}^k - \int_0^1 T(\tau)\,\mathrm{d}\tau \right\| |\hat{b}(z)| \to 0$$

as $n \to \infty$. *Thus*

$$\lim_{n\to\infty} (\widehat{a_n})\,(\xi) = \hat{a}(\xi), \quad \xi \in \Delta. \tag{12}$$

Observe that

$$(\widehat{a_n})(\xi) = z - (\hat{W}(\xi))^n - \frac{1}{n}\sum_{k=1}^{n} (\hat{W}(\xi))^k \tilde{b}(z).$$

Since the operator $T(1/n)$ *is compact, the Riesz-Schauder theorem implies that* $\sigma(T(1/n)) = P_\sigma(T(1/n)) \cup \{0\}$. *Also, it follows from Ref.* [9, *Theorems* 2.2.3–2.2.4] *that*

$$\exp\big((1/n)\sigma(A)\big) \subset \sigma(T(1/n)), \;\; P_\sigma(T(1/n)) \cup \{0\} = \exp\big((1/n)P_\sigma(A)\big) \cup \{0\}.$$

Therefore we get $\sigma(W) = \sigma(T(1/n)) = \exp((1/n)P_\sigma(A)) \cup \{0\}$. *By virtue of these observations, we see that the range of* $(\hat{a}_n)$ *is identical with the set*

$$\{z\} \cup \left(\{z - e^{\nu} - \frac{1}{n}\sum_{k=1}^{n} e^{(k/n)\nu}\tilde{b}(z) | \nu \in \sigma(A)\} \right).$$

Note that $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} e^{(k/n)\nu} = \int_0^1 e^{\nu\tau}\,\mathrm{d}\tau$. *Therefore, combining this fact with Eq.* (12) *we conclude that the set in the right hand side of Eq. (10) is identical with the range of* $\hat{a}$ *which is equal to* $\sigma(a) = \sigma(zI - \tilde{\mathrm{Q}}(z))$. *This completes the proof.* □

Observe that in the restricted case that $B(k) \equiv b(k)I$, the coefficients $Q = \{Q(n)\}$ in Eq. (8) satisfy the condition (i) in Theorem 1. Therefore, the following corollaries immediately follow from Theorems 2–4 and Proposition 1.

COROLLARY 2 *Let T(t) be a compact semigroup on X, and assume that* $B(k) \equiv b(k)I$, $k \in \mathbf{Z}^+$, *where b(k) is a scalar function satisfying* $\sum_{k=0}^{\infty}|b(k)| < \infty$. *Then the following two statements are equivalent:*

(i) The zero solution of Eq. (8) *is uniformly asymptotically stable;*
(ii) $z \neq e^{\nu} + \tilde{b}(z)\int_0^1 e^{\nu\tau}\,\mathrm{d}\tau, \quad (\forall\;\; |z| \geq 1, \;\; \nu \in \sigma(A))$.

COROLLARY 3 *Let all the conditions in Corollary* 2 *hold true, and assume that*

$$z \neq e^{\nu} + \tilde{b}(z)\int_0^1 e^{\nu\tau}\,\mathrm{d}\tau, \quad (\forall\;\; |z| \geq 1, \;\; \nu \in \sigma(A)).$$

Then the zero solution of Eq. (8) *is exponentially stable if and only if* $b(n)$ *decays exponentially.*

In the case where the dimension of X is finite, Theorem 2 and Corollary 1 remain valid without the condition (i) in Theorem 1, that is, the commutative condition on $Q(n)$ (cf. Ref. [4, Theorem 2]). On the one hand, in the case where the dimension of X is infinite, the result corresponding to those results can be established by imposing the condition that $Q(n)$ decays exponentially, instead of the commutative condition; see Ref. [5, Theorem 2]. In the case where the dimension of X is infinite, it is natural to ask if Theorem 2 and Corollary 1 in this paper remain valid without the commutative condition. Although the authors have not succeeded in answering the question generally, we can partly answer the question. Before concluding this paper, we will refer to this question in the following.

Let $\mathcal{A}$ be a commutative Banach algebra (containing the identity operator) in $\mathcal{L}(Y)$, where Y is a Banach space, and let us consider all of matrices whose components belong to $\mathcal{A}$. In what follows, we treat the space $M(\mathcal{A})$ of all 2×2 matrices, for simplicity. Each

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $M(\mathcal{A})$ may be considered as a bounded linear operator on the Banach space $X := Y \oplus Y$. We define the *determinant* $\det T$ of T by

$$\det T = ad - bc.$$

Of course, we get $\det T \in \mathcal{L}(Y)$. It is easy to see that if $\det T$ is invertible in $\mathcal{L}(Y)$, then T is invertible in $\mathcal{L}(X)$, and the inverse T^{-1} is given by

$$T^{-1} = \begin{pmatrix} d(\det T)^{-1} & -b(\det T)^{-1} \\ -c(\det T)^{-1} & a(\det T)^{-1} \end{pmatrix}.$$

Now, we consider Eq. (1) whose coefficients $Q(n)$ belong to $M(\mathcal{A})$. Notice that the condition (i) in Theorem 1 is not always satisfied. Let $R = \{R(n)\}$ be the fundamental solution of Eq. (1). It is easy to see that $R(n)$ belongs to $M(\mathcal{A})$. Assume that R is summable. Then, for any $|z| \geq 1$ we get $(zI - \tilde{\mathrm{Q}}(z))\tilde{\mathrm{R}}(z) = zI$, which yields that

$$\det(zI - \tilde{\mathrm{Q}}(z)) \cdot \det \tilde{\mathrm{R}}(z) = z^2 I.$$

Thus, if R is summable, then the following condition is satisfied;

(ii$*$) for any $|z| \geq 1$, $\det(zI - \tilde{\mathrm{Q}}(z))$ is invertible in $\mathcal{L}(Y)$.

Conversely, assume that the condition (ii$*$) is satisfied. Define $S = \{S(n)\}$ by the relation

$$S(0) = I, \quad S(n) = -Q(n-1) \quad n = 1, 2, \ldots,$$

as in the proof of Corollary 1. We claim that $\det(\sum_{k=0}^{\infty} S(k) w^k)$ is invertible in $\mathcal{L}(Y)$ for each $|w| \leq 1$. This claim follows from the condition (ii$*$), because of $z\tilde{S}(z) = zI - \tilde{\mathrm{Q}}(z)$ for $|z| \geq 1$. Observe that $f(w) := \det(\sum_{k=0}^{\infty} S(k) w^k)$ satisfies $f(0) = I$ and

$$f(w) = \tilde{a}(1/w)\tilde{d}(1/w) - \tilde{b}(1/w)\tilde{d}(1/w) = (\widetilde{a * d - b * c})(1/w)$$

for $0 < |w| \leq 1$, where

$$S(n) = \begin{pmatrix} a(n) & b(n) \\ c(n) & d(n) \end{pmatrix},$$

and $a = \{a(n)\}$, $b = \{b(n)\}$, $c = \{c(n)\}$ and $d = \{d(n)\}$. Combining the claim and Theorem 1, we see that $a * d - b * c \in L^1(\mathbf{Z}^+)$ is invertible in $L^1(\mathbf{Z}^+)$; that is, there exists an $r \in L^1(\mathbf{Z}^+)$ such that

$$\tilde{\mathrm{r}}(z) = [(\widetilde{a * d - b * c})(z)]^{-1} = [\det \tilde{S}(z)]^{-1}$$

for $|z| \geq 1$. Therefore it follows that

$$\left(I - \frac{1}{z}\tilde{\mathrm{Q}}(z)\right)^{-1} = (\tilde{S}(z))^{-1} = \begin{pmatrix} \tilde{d}(z)\,\tilde{\mathrm{r}}(z) & -\tilde{b}(z)\,\tilde{\mathrm{r}}(z) \\ -\tilde{c}(z)\,\tilde{\mathrm{r}}(z) & \tilde{a}(z)\,\tilde{\mathrm{r}}(z) \end{pmatrix}$$

for $|z| \geq 1$, which yields that

$$R(n) = \begin{pmatrix} (d*r)(n) & -(b*r)(n) \\ -(c*r)(n) & (a*r)(n) \end{pmatrix}$$

for $n \in \mathbf{Z}^+$. Consequently, $R = \{R(n)\}$ is summable.

Summarizing the above facts, we see that Corollary 1 remains valid (under the restricted situation) without the commutative condition if we replace the condition (ii$'$) by the condition (ii$*$). Similarly, we can remove the commutative condition in Theorem 2 if the condition (i) in Theorem 2 is replaced by the condition (ii$*$). We omit the details.

References

[1] L. A. V. Carvalho and K. L. Cooke, A nonlinear equation with piecewise continuous argument, *Differential and Integral Equations*, **1** (1988), 359–67.
[2] K. L. Cooke and J. Wiener, Retarded differential equations with piecewise constant delays, *J. Math. Anal. Appl.*, **99** (1984), 265–297.
[3] S. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.
[4] S. Elaydi and S. Murakami, Asymptotic stability versus exponential stability in linear Volterra difference equations of convolution type, *J. Difference Equations and Appl.*, **2** (1996), 401–410.
[5] T. Furumochi, S. Murakami, Y. Nagabuchi, Volterra difference equations on a Banach space and abstract differential equations with piecewise continuous delays, (submitted for publication).
[6] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers, 1992, Mathematics and its Applications.
[7] S. Mohamad and K. Gopalsamy, Extreme stability and almost periodicity in a discrete logistic equation, *Tohoku Math. J.*, **52** (2000), 107–125, Correction, *Tohoku Math. J.*, 53 (2001) 629–631.
[8] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991, Oxford Math. Monographs.
[9] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, **44**, Springer-Verlag, Berlin-New York, 1983, Applied Math. Sciences.
[10] W. Rudin, Functional Analysis, McGraw-Hill, New Delhi, 1988.
[11] J. Wiener, Generalized Solutions of Functional Differential Equations, World Scientific, Singapore, 1993.

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