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FINITE EXPONENTIAL SERIES APPROXIMATION OF DATA CURVE

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データ曲線の有限指数級数近似

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When empirical data correspond to a simple decay or growth process, or to a combination of these both processes, it is possible to approximate these data curve with a finite exponential series by Prony's method. This paper describes the numerical treatment of the Endochronic constitutive theory for soils which is the hereditary integral in form, and the method of the representation of finite exponential series of the integral kernel. Some examples of the approximation by finite exponential series for the error function and the normalized incomplete gamma function are given by means of Prony's method.

I. INTRODUCTION

The constitutive equation called the Endochronic theory in plasticity was introduced by Valanis in 1971, and has been received increasing attention as an alternate approach for describing the inelastic behavior of history-dependent materials. It is considered that the Endochronic theory in plasticity is divided into two types, that is, the integral form proposed by Valanis and the incremental form proposed by Bazant. The Endochronic approach differs substantially from classical plasticity and has many features which make attractive for modeling soil behavior.

This paper is devoted to the numerical treatment of the Endochronic theory for soils which introduces the concept of critical state soil mechanics. The constitutive equations have been expressed by the hereditary integral forms, therefore they are analytically complex and present serious difficulties from the numerical standpoint. The numerical treatment of the hereditary integral equations can be greatly simplified, if the kernel functions are approximated by finite series of exponentials. Although the resulting approximate kernels are no longer strictly singular from the mathematical standpoint, they can be made sufficiently singular for computational purposes.

When empirical data correspond to a simple decay or growth process, or to a combination of these both processes, and an approximation is desired for a semi-infinite range of the independent variable, the real exponential functions are appropriate coordinate functions from the point of view of representing the general data function by Dirichlet series. So that, in order to consider the representation of finite exponential series of the integral kernel in the Endochronic theory, some examples of the approximation by finite exponential series for the error function and the normalized incomplete gamma function are shown by Prony's method.

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II. PRONY'S METHOD[1]

We suppose here that a function $F(x)$ to be approximated is a data function, and in certain situation it is desired to determine an approximation of the form

$$F(x) \approx C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \cdots\cdots + C_n e^{\alpha_n x} \tag{1}$$

$$= C_1 \beta_1{}^x + C_2 \beta_2{}^x + \cdots\cdots + C_n \beta_n{}^x \tag{2}$$

where

$$\beta_k = e^{\alpha_k} (k=1, 2, \ldots\ldots, n). \tag{3}$$

We assume that values of $F(x)$ are specified on a set of N equally spaced points, and that a linear change of variables has been introduced in such a way that the data points are $x=0, 1, 2, \ldots\ldots, N\text{-}1$, i. e.,

$$\left.\begin{array}{l} C_1 + C_2 + \cdots\cdots + C_n = F_0 \\ C_1\beta_1 + C_2\beta_2 + \cdots\cdots + C_n\beta_n = F_1 \\ C_1\beta_1{}^2 + C_2\beta_2{}^2 + \cdots\cdots + C_n\beta_n{}^2 = F_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1\beta_1{}^{N-1} + C_2\beta_2{}^{N-1} + \cdots\cdots + C_n\beta_n{}^{N-1} = F_{N-1}. \end{array}\right\} \tag{4}$$

If the constants $\beta_1, \beta_2, \ldots\ldots, \beta_n$ were known, this set of equations would build up N linear equations in the n unknowns $C_1, C_2, \ldots\ldots, C_n$ and could be solved exactly if $N=n$ or approximately by the least-squares method if $N>n$.

On the other hand, if the β's are also to be determined, at least $2n$ equations are needed and the difficulty that the equations are nonlinear in the β's arises. To minimize this difficulty, let $\beta_1, \beta_2, \ldots\ldots, \beta_n$ be the roots of the following algebraic equation.

$$\beta^n + p_1\beta^{n-1} + p_2\beta^{n-2} + \cdots\cdots + p_{n-1}\beta + p_n = 0 \tag{5}$$

In this case, the left hand side of Eq.(5) is identified with the product

$$(\beta-\beta_1)(\beta-\beta_2)(\beta-\beta_3)\cdots\cdots(\beta-\beta_n).$$

In order to determine the coefficients $p_1, p_2, \ldots\ldots, p_n$, we multiply the first equation in (4) by p_n, the second equation by p_{n-1}, the third equation by $p_{n-2}, \ldots\ldots$, the n-th equation by p_1, and add the results. Using Eq.(5), the result is seen to be of the form

$$F_n + F_{n-1}p_1 + F_{n-2}p_2 + \cdots\cdots + F_0 p_n = 0.$$

A set of $N\text{-}n\text{-}1$ additional equations is obtained in the same way by starting instead successively with the second, third, $\ldots\ldots$, $(N\text{-}n)$th equations. In this way, we obtain the $N\text{-}n$ linear equations

$$\left.\begin{array}{l} F_n + F_{n-1}p_1 + F_{n-2}p_2 + \cdots\cdots + F_0 p_n = 0 \\ F_{n+1} + F_n p_1 + F_{r-1}p_2 + \cdots\cdots + F_1 p_n = 0 \\ F_{n+2} + F_{n+1}p_1 + F_n p_2 + \cdots\cdots + F_2 p_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{N-1} + F_{N-2}p_1 + F_{N-3}p_2 + \cdots\cdots + F_{N-n-1}p_n = 0. \end{array}\right\} \tag{6}$$

Since the values F_k $(k=0, 1, 2, \ldots\ldots, N-1)$ are known, the set of these equations generally can be solved directly for the n p's if $N=2n$, or solved approximately by the least-squares method if $N>2n$.

After the n p's are determined, the n β's are found as the roots of Eq. (5). These roots may be real or imaginary. Then Eqs. (4) become linear equations in the n C's with known coefficients. The C's can be determined, finally from the first n of these equations, or preferably by applying the least-squares method to the whole equations. So that, the nonlinearity of the system of Eqs. (4) is reduced to the single algebraic equation (5).

III. REDUCTION OF THE ENDOCHRONIC SOIL MODEL TO DIFFERENTIAL EQUATIONS

The broadest statement of the Endochronic theory in plasticity is that the state of stress of a material element in its present configuration is a function of the history of deformation of the element with respect to the intrinsic time (internal time). This idea was introduced by Valanis, and was recently modified by Read[2] for describing the nonlinear, inelastic behavior of soils. The constitutive equations which define the Endochronic soil model are as follows,

$$\sigma=H(v,\ v_0)\int_{-\infty}^{z_H}\phi(z_H-z')\frac{d\varepsilon^p}{dz'}dz' \tag{7}$$

$$\underset{\sim}{s}=F(\sigma,\ v)\int_0^{z_D}\rho(z_D-z')\frac{\partial \underset{\sim}{e}^p}{\partial z'}dz' \tag{8}$$

where $\sigma=\sigma_{kk}$=hydrostatic stress, $\underset{\sim}{s}$=deviatoric stress tensor, $\varepsilon^p=\varepsilon_{kk}{}^p$=plastic volumetric strain, $\underset{\sim}{e}^p$=plastic deviatoric strain tensor, z_H and z_D denote the hydrostatic intrinsic time and the deviatoric intrinsic time respectively, H is a function of the current specific volume v and the initial specific volume v_0 which describes the state on which the current hydrostatic state is located, and F is a function of σ and v which describes the effect of the state of the material on the deviatoric stress.

However the above equations for the Endochronic soil model are analytically complex and involve difficulties in numerical treating due to the hereditary integral expression for σ and $\underset{\sim}{s}$.

If the kernel functions are approximated by finite exponential series, the numerical treatment of the hereditary integrals can be simplified. It is indicated by Read[2] that for soils an adequate representation can be achieved with several terms in a series, and that only three terms in a series are used in most cases. In this case, the hereditary integrals can be reduced to differential constitutive equations. We adopt the following finite exponential series of the kernel functions $\phi(z)$ and $\rho(z)$,

$$\phi(z)=\sum_{r=1}^{m}P_r\cdot e^{-\beta_r z} \tag{9}$$

$$\rho(z)=\sum_{r=1}^{n}R_r\cdot e^{-\alpha_r z} \tag{10}$$

which satisfy the conditions

$$\phi(0)=\infty,\ \int_0^{\infty}\phi(z')dz'=1 \tag{11}$$

$$\rho(0)=\infty,\ \int_0^{\infty}\rho(z')dz'=1 \tag{12}$$

where P_r, β_r, R_r and α_r are positive constants.

Now, when we use the forms of $\phi(z)$ and $\rho(z)$, given by Eqs. (9) and (10), the hereditary integrals of Eqs. (7) and (8) can be reduced to the following differential equations.

$$\sigma=H\sum_{r=1}^{m}S_r,\ \frac{dS_r}{dz_H}+\beta_r S_r=P_r\frac{d\varepsilon^p}{dz_H} \tag{13}$$

$$\underset{\sim}{s}=F\sum_{r=1}^{n}\underset{\sim}{Q}_r,\ \frac{d\underset{\sim}{Q}_r}{dz_D}+\alpha_r\underset{\sim}{Q}_r=R_r\frac{d\underset{\sim}{e}^p}{dz_D} \tag{14}$$

In case of the deviatoric response, for example, Eq. (14) describes the mechanical model response of the parallel assembly model of endochronic elements shown in Fig. 1, when b_r

and R_r are constants. An endochronic element model[4], as shown in Fig. 1, can be constracted by connecting, in series, a linear elastic spring and a nonlinear endochronic slider. In this figure, R_r and b_r denote the spring constant and the slider resistance respectively. And also $\underset{\sim}{Q}_r$ denotes the stress of the r-th endochronic element. A similar mechanical model can be constructed for the hydrostatic component of response given by Eq. (13).

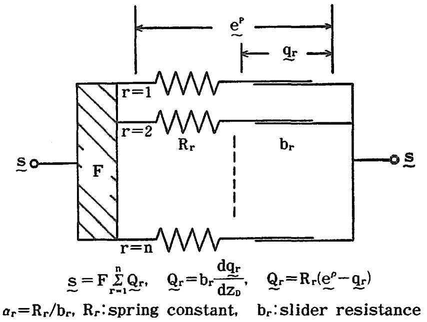


Fig. 1 A parallel assembly of endochronic elements.

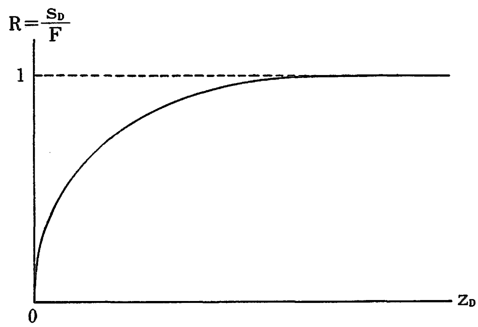


Fig. 2 Normalized data curve from triaxial compression tests for various confining pressures.

IV. DETERMINATION OF THE KERNEL FUNCTIONS

To determine specific forms of the kernel functions, $\phi(z)$ and $\rho(z)$, we consider a conventional triaxial compression test, in which the axial stress and the axial strain can be denoted by σ_1 and ε_1 respectively. In this case, a soil element experiences a loading in which the deviatoric stress tensor $\underset{\sim}{s}$ and the deviatoric plastic strain tensor $\underset{\sim}{e}^p$ can be expressed as follows,

$$\underset{\sim}{s}=s_D\cdot\underset{\sim}{t},\ s_D=\sqrt{\frac{3}{2}}s_1 \tag{15}$$

$$\underset{\sim}{e}^p=z_D\cdot\underset{\sim}{t},\ z_D=\sqrt{\frac{3}{2}}e_1{}^p \tag{16}$$

and $\underset{\sim}{t}$ is a constant direction unit tensor which can be expressed in the following matrix form.

$$\underset{\sim}{t}=\frac{1}{\sqrt{6}}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix} \tag{17}$$

We consider the deviatoric response especially, then Eq. (8) can be written in the form for monotonic loading by using Eq. (15).

$$\frac{s_D}{F}=\int_0^{z_D}\rho(z')dz'\equiv R(z_D) \tag{18}$$

And the integral function $R(z_D)$ must satisfy the condition,

$$\lim_{z_D\to\infty}R(z_D)=1. \tag{19}$$

The relation between s_D and z_D is shown as a curve for various lateral stress σ_3 using the triaxial compression data, and all of these curves of s_D/F vs z_D may be ideally rearranged as a single curve having the general form shown in Fig. 2 by suitable choice of the form of the function F and approximate representation of the kernel function $\rho(z)$. To obtain the approximate representation of $\rho(z)$ by a finite exponential series, we use Eqs. (10) and (18), and obtain the following expression.

$$R(z_D)=\sum_{r=1}^{n} R_r\int_0^{z_D} e^{-\alpha_r z'}\,dz'=\sum_{r=1}^{n}\frac{R_r}{\alpha_r}(1-e^{-\alpha_r z_D}) \tag{20}$$

To satisfy the condition of Eq. (19), it is necessary to impose the condition,

$$\sum_{r=1}^{n}\frac{R_r}{\alpha_r}=1. \tag{21}$$

From Eqs. (20) and (21), the exponential series representation of $R(z_D)$ is as follows.

$$R(z_D)=1-\sum_{r=1}^{n}\frac{R_r}{\alpha_r}e^{-\alpha_r z_D} \tag{22}$$

From Eq. (22), the representation of $1-R(z_D)$ by a finite exponential series can be accomplished by Prony's method described in Section II.

It was shown by Read[2] that the function $R(z_D)$ can be well represented for soils by the normalized incomplete gamma function $\gamma(k\,z_D,\ a)$, that is,

$$R(z_D)=\gamma(k\,z_D,\ a)=\frac{1}{\Gamma(a)}\int_0^{kz_D} e^{-t}\cdot t^{a-1}\,dt \quad (0<a<1) \tag{23}$$

where k and a are positive constants, $\Gamma(a)$ denotes the complete gamma function. Once the constants k and a have been determined through the least-squares method by fitting to the normalized data such as given in Fig. 2, the normalized incomplete gamma function $\gamma(k\,z_D,\ a)$ is approximated by a finite exponential series through Eq. (22). That means, since $1-\gamma(k\,z_D,\ a)$ is a completely monotonic function of z_D, Prony's method is guaranteed to yield positive decay exponents α_r and positive coefficients R_r. Because Prony's method consists of interpolation of the finite exponential series at equidistance points, the choice $z_D=0$, as one of the interpolation points, guarantees that Eq. (21) will be satisfied.

And in much the same way as the deviatoric response, in case of the hydrostatic response, it was shown by Read[2] that the following function $P(z_H)$ can be represented by the error function $erf\,(\sqrt{k\,z_H})$.

$$P(z_H)=\frac{\sigma}{H}=\int_0^{z_H}\phi(z')\,dz'=erf\,(\sqrt{k\,z_H}) \tag{24}$$

The error function is then represented by a finite exponential series in the manner of using Eqs. (9) and (24),

$$1-erf\,(\sqrt{k\,z_H})=\sum_{r=1}^{m}\frac{P_r}{\beta_r}e^{-\beta_r z_H} \tag{25}$$

where the constants P_r and β_r can be determined by Prony's method.

V. EXAMPLES OF APPLICATION OF PRONY'S METHOD AND CONCLUSIONS

By using Prony's method, we consider the following approximation by finite exponential series,

$$1-\gamma(k\,z_D,\ a)=\sum_{r=1}^{n}\frac{R_r}{\alpha_r}e^{-\alpha_r z_D} \tag{26}$$

$$1-erf\,(\sqrt{k\,z_H})=\sum_{r=1}^{m}\frac{P_r}{\beta_r}e^{-\beta_r z_H}, \tag{27}$$

that is, when the parameters k and a in the left hand side of Eqs. (26) and (27) are prescribed, let us determine positive constants R_r, α_r, P_r and β_r in the right hand side of Eqs. (26) and(27).

Now for example, we assume the functions given in the left hand side of Eqs. (26) and (27), with $k=25$ and $a=0.3$, and fit the functions to the three or fewer terms decaying exponential series of the forms of Eqs. (26) and (27) by Prony's method.

Firstly, in the case of the function given in the left hand side of Eq. (26), with $k=25$

and $a=0.3$, the approximation by finite exponential series is shown in Fig. 3 for various sampling intervals of equidiatant points. And the values of exponents and coefficients in the series obtained in each case are shown in Table 1.

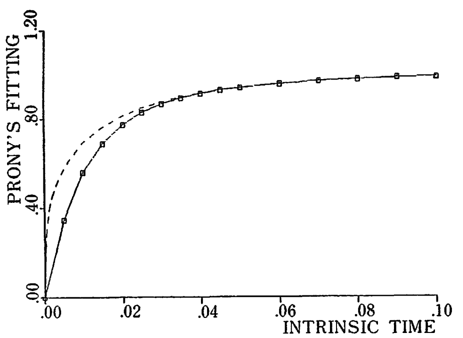
Secondly, in the case of the function given in the left hand side of Eq. (27), with $k=25$, the approximation by finite exponential series is shown in Fig. 4 for various sampling intervals of equidistant points and various numbers of terms in the series. And the values of exponents and coefficients in the series obtained in each case are shown in Table 2.

The major conclusions that have come out of the present study are as follows :

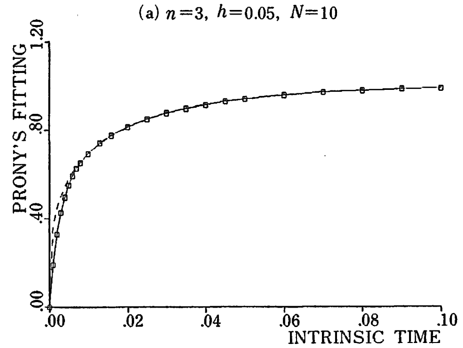
(1) An adequate representation can be achieved with three or fewer terms in a series, and in most cases, it is sufficient to use only three terms for such a completely monotonic function.

(2) With regard to sampling interval for equidistant points of the data function, it seems to be necessary to take small sampling interval for interpolation of the exponential series in the range of the steep slope of the function. However, it is not always necessary to take many points of evaluation, and it is sufficient to take points about twice as many as the number of terms in the series.

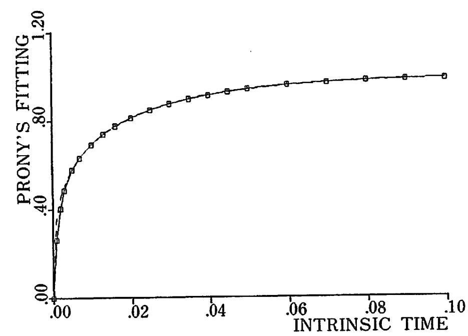
(3) The value of finite exponential series should be so large at zero point of intrinsic time that it is effectively singular.



(a) $n=3$, $h=0.05$, $N=10$



(b) $n=3$, $h=0.01$, $N=10$



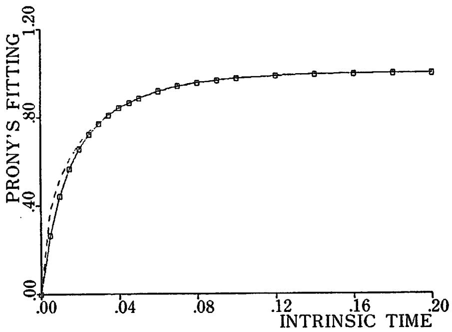
(c) $n=3$, $h=0.005$, $N=10$

Fig. 3 Fitting of the normalized incomplete gamma function to finite exponential series by Prony's method (n : number of terms in series, h : sampling interval for equidistant points, N: number of points of evaluation, ------- data function, —⊟— Prony series)

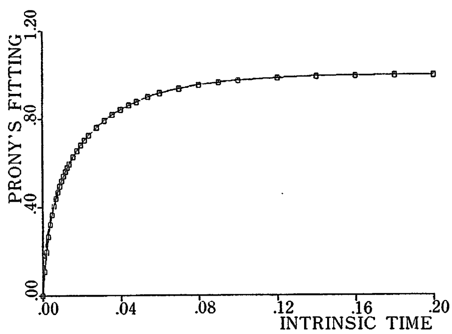
Table 1 Values of exponent α_r and coefficient R_r for the normalized incomplete gamma function

Term (r)	$n=3$, $h=0.05$, $N=10$		$n=3$, $h=0.01$, $N=10$		$n=3$, $h=0.005$, $N=10$	
	R_r	α_r	R_r	α_r	R_r	α_r
1	7.87×10	1.13×10^2	2.02×10^2	4.04×10^2	3.16×10^2	7.51×10^2
2	6.97	4.04×10	1.95×10	8.40×10	3.10×10	1.31×10^2
3	3.48	2.69×10	8.38	3.14×10	1.21×10	3.54×10

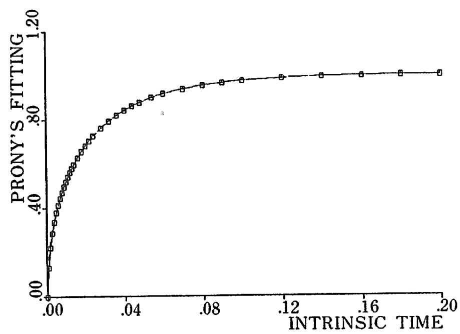
n : number of terms, h : sampling interval, N: number of points of evaluation



(a) $m=3$, $h=0.05$, $N=10$



(b) $m=3$, $h=0.01$, $N=10$



(c) $m=5$, $h=0.01$, $N=10$

Fig. 4 Fitting of the error function to finite exponential series by Prony's method (m : number of terms in series, h : sampling interval for equidistant points, N : number of points of evaluation, ------ data function, —□— Prony series)

Table 2 Values of exponent β_r and coefficient P_r for the error function

Term (r)	$m=3$, $h=0.05$, $N=10$		$m=3$, $h=0.01$, $N=10$		$m=5$, $h=0.01$, $N=10$	
	P_r	β_r	P_r	β_r	P_r	β_r
1	4.73×10	9.97×10	9.25×10	3.41×10^2	1.08×10^2	5.92×10^2
2	9.41	3.88×10	1.99×10	7.63×10	1.93×10	2.05×10^2
3	7.44	2.63×10	1.37×10	2.93×10	1.34×10	9.62×10
4	—	—	—	—	1.09×10	4.68×10
5	—	—	—	—	9.56	2.72×10

m : number of terms, h : sampling interval, N : number of points of evaluation

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