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ON THE DIMENSIONS OF VECTOR SPACES CONCERNING HOLOMORPHIC VECTOR BUNDLES OVER ELLIPTIC ORBITS, II

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Abstract. This paper deals with the complex vector space of holomorphic cross-sections of a homogeneous holomorphic vector bundle over an elliptic adjoint orbit of a connected real simple Lie group of Hermitian type, and provides a necessary and sufficient condition for the vector space to be finite-dimensional.

Contents

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1. Introduction and the main result (Theorem 1.2)

This is a sequel to the paper [3] where we study the complex vector space $\mathcal{V}_{G/L}$ of holomorphic cross-sections of a homogeneous holomorphic vector bundle over an elliptic (adjoint) orbit G/L of a connected real semisimple Lie group G and give a sufficient condition for the vector space $\mathcal{V}_{G/L}$ to be finite-dimensional. First of all, let us recall a result in [3]. Let $G_{\mathbb{C}}$ be a connected complex semisimple Lie group, G a connected closed subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$, and T a non-zero elliptic element of $\mathfrak{g}$. Setting

$$\begin{aligned}&L := C_G(T), \quad \mathfrak{g}^{\lambda} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\} \text{ for a } \lambda \in \mathbb{R},\\ &Q^- := N_{G_{\mathbb{C}}}(\textstyle\bigoplus_{\mu \geq 0} \mathfrak{g}^{-\mu}),\end{aligned}$$

one has an elliptic orbit G/L, a complex parabolic subgroup $Q^- \subset G_{\mathbb{C}}$ and a complex flag manifold $G_{\mathbb{C}}/Q^-$ (which is also called a Kähler C-space or a generalized flag manifold), and knows that $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, is a G-equivariant real analytic diffeomorphism of G/L onto a domain in $G_{\mathbb{C}}/Q^-$. Then we identify G/L with its image $\iota(G/L)$, induce the complex structure J on G/L from $G_{\mathbb{C}}/Q^-$, and consider $G/L = (G/L, J)$ as a homogeneous complex manifold of G.

$$G/L \xrightarrow{\ \iota\ } G_{\mathbb{C}}/Q^-$$

For any finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$, we denote by $G_{\mathbb{C}} \times_{\rho} \mathsf{V}$ the homogeneous holomorphic vector bundle over the complex flag manifold $G_{\mathbb{C}}/Q^-$ associated with ρ, by $\iota^{\sharp}(G_{\mathbb{C}} \times_{\rho} \mathsf{V})$ its restriction to the domain $G/L = \iota(G/L) \subset G_{\mathbb{C}}/Q^-$, and by $\mathcal{V}_{G/L}$ or $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$ the complex vector space of holomorphic cross-sections of the bundle $\iota^{\sharp}(G_{\mathbb{C}} \times_{\rho} \mathsf{V})$.

$$\begin{array}{ccc}\iota^{\sharp}(G_{\mathbb{C}} \times_{\rho} \mathsf{V}) & & G_{\mathbb{C}} \times_{\rho} \mathsf{V}\\ \downarrow & & \downarrow\\ G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^-\end{array}$$

In this setting, we have established

Theorem 1.1 (cf. Theorem 1.0.1 in [3, p.219])**.** $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$ *if a maximal compact subalgebra of* $\mathfrak{g}$ *is semisimple* (*more precisely,* $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ *for every finite-dimensional complex vector space* V *and holomorphic homomorphism* $\rho : Q^- \to GL(\mathsf{V})$, *if a maximal compact subalgebra of* $\mathfrak{g}$ *is semisimple*).

Our goal is to give a necessary and sufficient condition for $\mathcal{V}_{G/L}$ to be finite-dimensional. Theorem 1.1 enables us to achieve this goal in the case where $\mathfrak{g}$ is a simple Lie algebra, provided that we demonstrate the following theorem which is the main result in this paper (see Remark 1.3-(i)):

Theorem 1.2. *In the same setting as that of Theorem* 1.1*, suppose* $\mathfrak{g}$ *to be a simple Lie algebra of Hermitian type. Then,* $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ *for every finite-dimensional complex vector space* V *and holomorphic homomorphism* $\rho : Q^- \to GL(\mathsf{V})$ *if and only if all holomorphic functions on* G/L *are constant.*

The simple Lie algebras of Hermitian type	
(AIII) $\mathfrak{su}(p,q)$ with $p,q \geq 1$	(BDI) $\mathfrak{so}(2,n)$ with $n \geq 4$
(DIII) $\mathfrak{so}^*(2m)$ with $m \geq 3$	(CI) $\mathfrak{sp}(p,\mathbb{R})$ with $p \geq 1$
(EIII) $\mathfrak{e}_{6(-14)}$	(EVII) $\mathfrak{e}_{7(-25)}$

Remark 1.3. Here are comments on Theorems 1.1 and 1.2.

(i) In the same setting as that of Theorem 1.1, suppose that $\mathfrak{g}$ is a simple Lie algebra. Then, $\mathfrak{g}$ is of Hermitian type if and only if a maximal compact subalgebra of $\mathfrak{g}$ is not semisimple. Accordingly Theorems 1.2 and 1.1 imply that the following conditions (A) and (B) are equivalent, where we remark that

$$\dim_{\mathbb{C}} \mathcal{O}(G/L) = 1$$

whenever a maximal compact subalgebra of $\mathfrak{g}$ is semisimple (cf. Corollary 3.4.6-(ii) in [3, p.248]):

(A) $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ for every finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$.

(B) $\dim_{\mathbb{C}} \mathcal{O}(G/L) = 1$.

Here $\mathcal{O}(G/L)$ stands for the complex vector space of holomorphic functions on G/L.

(ii) In the same setting as that of Theorem 1.1, there always exist connected complex semisimple Lie groups $(G_a)_{\mathbb{C}}$, connected closed subgroups $G_a \subset (G_a)_{\mathbb{C}}$ and non-zero elliptic elements $T_a \in \mathfrak{g}_a$ such that each $\mathfrak{g}_a$ is a real form of $(\mathfrak{g}_a)_{\mathbb{C}}$ and a simple Lie algebra,[1] and that G/L is G-equivariant biholomorphic to the direct product $G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$ where $L_a := C_{G_a}(T_a)$ and we fix a complex structure J_a on G_a/L_a in a way similar to the way of fixing the J on G/L $(1 \leq a \leq n)$. However, the author does not know whether the following conditions (α) and (β) are equivalent, or more precisely, whether (β) implies (α):

(α) $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ for every finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$.

(β) $\dim_{\mathbb{C}} \mathcal{O}(G_a/L_a) = 1$ for each direct factor G_a/L_a of the $G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$.

(iii) One might derive a more general statement than our Theorems 1.1 and 1.2 from the two proofs of Proposition 2.5 in Dunne-Zierau [5, p.493] and Proposition 3.16 in Zierau [18, p.115], or from Theorem 3.1 and Proposition 2.1 in Huckleberry [8, p.118, p.117]. Here, the proof of Dunne-Zierau [5, p.493] depends on results in Wong [17], Vogan [16] and Harish-Chandra [6], and the proof of Huckleberry [8, p.118, p.117] depends on results in the study on cycle spaces of flag domains. Our proof of Theorem 1.2 is independent of the results, but we complete the proof with referring to the two proofs of Dunne-Zierau [5, p.493] and Zierau [18, p.115].

[1] Although $\mathfrak{g}_a$ is simple, $(\mathfrak{g}_a)_{\mathbb{C}}$ is not necessarily simple.

This paper consists of three sections. In §2 we recall the definition of elliptic orbit, show some lemmas and collect former results. In §3 we establish some propositions and finally conclude Theorem 1.2 by them.

Notation. Throughout this paper, for a Lie group G we denote its Lie algebra by the corresponding Fraktur small letter $\mathfrak{g}$ and use the following notation:

(n1) $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$: the sets of natural numbers, integers, real numbers, and complex numbers, respectively, where $\mathbb{N}$ does not contain the zero,
(n2) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,
(n3) $GL(\mathsf{V})$, $\mathfrak{gl}(\mathsf{V})$: the general linear group, and linear Lie algebra on a complex vector space V, respectively,
(n4) Ad, ad : the adjoint representations of G and $\mathfrak{g}$, respectively,
(n5) $C_G(\mathfrak{t}) := \{g \in G \mid \operatorname{Ad} g(H) = H \text{ for all } H \in \mathfrak{t}\}$ for a subset $\mathfrak{t} \subset \mathfrak{g}$,
(n6) $C_G(T) := \{g \in G \mid \operatorname{Ad} g(T) = T\}$ for an element $T \in \mathfrak{g}$,
(n7) $N_G(\mathfrak{m}) := \{g \in G \mid \operatorname{Ad} g(\mathfrak{m}) \subset \mathfrak{m}\}$ for a vector subspace $\mathfrak{m} \subset \mathfrak{g}$,
(n8) $B_{\mathfrak{g}}$: the Killing form of $\mathfrak{g}$,
(n9) $f|_S$: the restriction of a mapping f to a set S,
(n10) $\mathbb{C}^* := \mathbb{C} - \{0\}$,
(n11) $i := \sqrt{-1}$.

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2. Preliminaries

This section consists of five subsections. In §§2.1 we recall the definitions of elliptic element and elliptic (adjoint) orbit. In §§2.2 we set an elliptic orbit G/L and a complex flag manifold $G_{\mathbb{C}}/Q^-$. In §§2.3 we set vector bundles $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and $G_{\mathbb{C}} \times_\rho \mathsf{V}$ over the elliptic orbit G/L and the complex flag manifold $G_{\mathbb{C}}/Q^-$, respectively, and set a continuous representation ϱ (resp. $\tilde{\varrho}$) of the Lie group G (resp. $G_{\mathbb{C}}$) on the vector space $\mathcal{V}_{G/L}$ (resp. $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$) of cross-sections of the bundle $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ (resp. $G_{\mathbb{C}} \times_\rho \mathsf{V}$). Thereafter, we construct arguments in their setting. In §§2.4 we show four lemmas. Finally in §§2.5 we collect former results, Theorem 2.17, Lemma 2.18, Lemma 2.19. The six lemmas and the theorem will play important parts in the next section.

2.1. Definition of elliptic orbit.

Here are the definitions of elliptic element and elliptic (adjoint) orbit.

Definition 2.1 (cf. Kobayashi [9])**.** Let $\mathfrak{g}$ be a real semisimple Lie algebra, and G a connected Lie group with Lie algebra $\mathfrak{g}$.

(i) An element $T \in \mathfrak{g}$ is said to be *elliptic*, if the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto [T, X]$, is semisimple and all the eigenvalues of $\operatorname{ad} T$ are purely imaginary.
(ii) The adjoint orbit $\operatorname{Ad} G(T) = G/C_G(T)$ of G through an elliptic element $T \in \mathfrak{g}$ is called an *elliptic adjoint orbit* or an *elliptic orbit* for short.

2.2. **Setting on an elliptic orbit and a complex flag manifold.** Hereafter, we obey the following setting which is a little different from the setting of [3, pp.223–224]:

- $G_{\mathbb{C}}$ is a connected complex semisimple Lie group,
- G is a connected closed subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$,
- T is a non-zero elliptic element of $\mathfrak{g}$,
- $L := C_G(T)$, $L_{\mathbb{C}} := C_{G_{\mathbb{C}}}(T)$, $\mathfrak{g}^{\lambda} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\}$ for a $\lambda \in \mathbb{R}$,
- $\mathfrak{u}^s := \bigoplus_{\lambda>0} \mathfrak{g}^{s\lambda}$, $U^s := \exp \mathfrak{u}^s$ and $Q^s := N_{G_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^s)$ for each $s = \pm$,
- θ is a Cartan involution of $\mathfrak{g}$ satisfying $T = \theta(T)$,[2]
- $\mathfrak{k} := \{X \in \mathfrak{g} \mid X = \theta(X)\}$, $\mathfrak{p} := \{Y \in \mathfrak{g} \mid Y = -\theta(Y)\}$, $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$,
- K and G_u are the maximal compact subgroups of G and $G_{\mathbb{C}}$ whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{g}_u$, respectively,
- $L_u := C_{G_u}(T)$,
- $\mathfrak{t}$ is a maximal torus of $\mathfrak{k}$ containing the T,
- $\mathfrak{h} := \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$, $\mathfrak{a} := \mathfrak{p} \cap \mathfrak{h}$, $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{t} \oplus \mathfrak{a}$,
- $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{t}_{\mathbb{C}}$, $\mathfrak{h}_{\mathbb{C}}$, and $\mathfrak{p}_{\mathbb{C}}$ are the complex subalgebras, and vector subspace of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{k}$, $\mathfrak{t}$, $\mathfrak{h}$, and $\mathfrak{p}$, respectively,
- $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the non-zero root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$,
- $\mathfrak{g}_{\alpha}$ is the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for an $\alpha \in \triangle$,
- $\{E_{\alpha}\}_{\alpha\in\triangle}$ is Chevalley's canonical basis of $\mathfrak{g}_{\mathbb{C}}$ mod $\mathfrak{h}_{\mathbb{C}}$ such that $(E_{\alpha} - E_{-\alpha})$, $i(E_{\alpha} + E_{-\alpha}) \in \mathfrak{g}_u$ for all $\alpha \in \triangle$,[3] and

for an $H \in \mathfrak{t}$ we put

$$
\begin{aligned}
&\triangle(H,0) := \{\gamma \in \triangle \mid \gamma(H) = 0\}, \quad \triangle(H,+) := \{\beta \in \triangle \mid \beta(-iH) > 0\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \triangle(H,-) := \{\beta \in \triangle \mid \beta(-iH) < 0\}.
\end{aligned}
\tag{2.2}
$$

In addition to this setting, we suppose $\triangle(T,+)$ to consist of N-roots $\beta_1, \beta_2, \dots, \beta_N$. Note here that

(1) the centers of $G_{\mathbb{C}}$ and G are both finite,
(2) both K and G_u are connected,
(3) $\mathfrak{l}$ and $\mathfrak{l}_{\mathbb{C}}$ are reductive subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$, respectively,
(4) $\mathfrak{h}_{\mathbb{R}} = \{A \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(A) \in \mathbb{R}$ for all $\alpha \in \triangle\}$,
(5) $\mathfrak{g}_{\alpha} = \operatorname{span}_{\mathbb{C}}\{E_{\alpha}\}$ for all $\alpha \in \triangle$,
(6) $\mathfrak{h}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{H^*_{\alpha}\}_{\alpha\in\triangle}$, $[H^*_{\alpha}, E_{\alpha}] = 2E_{\alpha}$ and $[H^*_{\alpha}, E_{-\alpha}] = -2E_{-\alpha}$ if one puts $H^*_{\alpha} := [E_{\alpha}, E_{-\alpha}]$ for each $\alpha \in \triangle$,
(7) $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\lambda\in\mathbb{R}} \mathfrak{g}^{\lambda} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^- = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\triangle} \mathfrak{g}_{\alpha}$,
(8) $\mathfrak{g}_u = i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\triangle} \operatorname{span}_{\mathbb{R}}\{E_{\alpha} - E_{-\alpha}\} \oplus \operatorname{span}_{\mathbb{R}}\{i(E_{\alpha} + E_{-\alpha})\}$,
(9) $\mathfrak{l}_{\mathbb{C}} = \mathfrak{g}^0 = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\gamma\in\triangle(T,0)} \mathfrak{g}_{\gamma}$,
(10) $\mathfrak{u}^+ = \bigoplus_{\lambda>0} \mathfrak{g}^{\lambda} = \bigoplus_{\beta\in\triangle(T,+)} \mathfrak{g}_{\beta} = \bigoplus_{\ell=1}^{N} \mathfrak{g}_{\beta_{\ell}}$,
(11) $\triangle(T,\pm) = -\triangle(T,\mp)$,

[2] Such a θ always exists (e.g. Lemma 7.2.4 in [2, p.69]).

[3] Such a $\{E_{\alpha}\}_{\alpha\in\triangle}$ exists, since $i\mathfrak{h}_{\mathbb{R}}$ is a maximal torus of $\mathfrak{g}_u$ (cf. Lemma 3.1 in Helgason [7, pp.257–258]).

and that $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$ if and only if $\mathfrak{g}$ has a compact Cartan subalgebra. In the setting above, we show two Lemmas 2.3 and 2.5.

Lemma 2.3. *There exists a fundamental root system* Π_1 *of* $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ *satisfying*

$$\alpha(-iT) \geq 0 \text{ for all } \alpha \in \Pi_1. \tag{2.4}$$

Proof. One can obtain such a Π_1 by defining the lexicographic linear ordering on the dual space $(\mathfrak{h}_{\mathbb{R}})^*$ associated with an ordered real basis $-iT =: A_1, A_2, \ldots, A_l$ of $\mathfrak{h}_{\mathbb{R}}$, for example. □

Lemma 2.5. *The following nine items hold*:

(i) *The closed subgroup* $L \subset G$ *is connected.*
(ii) *The closed complex* (*Lie*) *subgroup* $L_{\mathbb{C}} \subset G_{\mathbb{C}}$ *is connected.*
(iii) U^s *is a simply connected, closed complex nilpotent subgroup of* $G_{\mathbb{C}}$ *whose Lie algebra coincides with* $\mathfrak{u}^s$, *and* $\exp : \mathfrak{u}^s \to U^s$ *is biholomorphic* $(s = \pm)$.
(iv) Q^s *is a connected, closed complex parabolic subgroup of* $G_{\mathbb{C}}$ *such that* $Q^s = L_{\mathbb{C}} \ltimes U^s$ (*semidirect*) *and* $\mathfrak{q}^s = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^s = \bigoplus_{\mu \geq 0} \mathfrak{g}^{s\mu}$ $(s = \pm)$.
(v) *The product mapping* $U^+ \times Q^- \ni (u, q) \mapsto uq \in G_{\mathbb{C}}$ *is a biholomorphism of* $U^+ \times Q^-$ *onto a domain in* $G_{\mathbb{C}}$.
(vi) $L_u = G_u \cap Q^s$, $L = G \cap Q^s$ $(s = \pm)$.
(vii) $\iota_u : G_u/L_u \to G_{\mathbb{C}}/Q^-$, $kL_u \mapsto kQ^-$, *is a* G_u*-equivariant real analytic diffeomorphism of* G_u/L_u *onto* $G_{\mathbb{C}}/Q^-$.
(viii) $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, *is a* G*-equivariant real analytic diffeomorphism of* G/L *onto a simply connected domain in* $G_{\mathbb{C}}/Q^-$.
(ix) GQ^- *is a domain in* $G_{\mathbb{C}}$.

Proof. We conclude this lemma by taking a fundamental root system $\Pi_1 \subset \triangle$ with (2.4) and the proofs of Lemma 7.3.3, Lemma 8.0.1, Proposition 8.2.1 and Lemma 11.1.2 in [2, p.71, p.75, p.78, p.117] into consideration. □

In general, there are several kinds of invariant complex structures on an elliptic orbit (see Example 2.6 below). Identifying the elliptic orbit G/L with the domain $\iota(G/L)$ in the complex flag manifold $G_{\mathbb{C}}/Q^-$ via $\iota : gL \mapsto gQ^-$, we fix an invariant complex structure J on G/L in the same way as we did in [3, Remark 2.4.4, p.225].[4]

$$G/L \xrightarrow{\quad \iota \quad} G_{\mathbb{C}}/Q^-$$

Example 2.6. Let $G = SL(2, \mathbb{C})$ and

$$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then it follows that $T \in \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ is represented by

$$\operatorname{ad} T = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[4]Note here that $N = \dim_{\mathbb{C}} \mathfrak{u}^+ = \dim_{\mathbb{C}} G/L = \dim_{\mathbb{C}} G_{\mathbb{C}}/Q^-$.

relative to a basis $\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$ of $\mathfrak{g}$. Hence T is a non-zero elliptic element of $\mathfrak{g}$. A direct computation yields

$$L = C_G(T) = \left\{\begin{pmatrix}a&0\\0&1/a\end{pmatrix} \,\middle|\, a\in\mathbb{C}^*\right\} = S(GL(1,\mathbb{C})\times GL(1,\mathbb{C})),$$

and one has $G/L = SL(2,\mathbb{C})/S(GL(1,\mathbb{C})\times GL(1,\mathbb{C}))$. Since this elliptic orbit is a reductive homogeneous space, especially an affine symmetric space, one can construct G-invariant complex structures J_1 and J_2 on G/L from

$$\mathrm{J}_1 : \begin{pmatrix}0&z\\w&0\end{pmatrix}\mapsto\begin{pmatrix}0&iz\\iw&0\end{pmatrix},\quad \mathrm{J}_2 : \begin{pmatrix}0&z\\w&0\end{pmatrix}\mapsto\begin{pmatrix}0&iz\\-iw&0\end{pmatrix}\quad\text{for } z,w\in\mathbb{C},$$

respectively. In this paper, we adopt the J_2 as a complex structure on $G/L = SL(2,\mathbb{C})/S(GL(1,\mathbb{C})\times GL(1,\mathbb{C}))$. Incidentally, G/L is a Stein manifold with respect to the J_1. cf. Théorèmes 5 et 1 dans Matsushima-Morimoto [13, p.151, p.139].

2.3. **Setting on vector bundles over G/L and $G_{\mathbb{C}}/Q^-$.** To the setting of §§2.2, we add the following setting. For any finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^-\to GL(\mathsf{V})$, $q\mapsto\rho(q)$, we denote by $G_{\mathbb{C}}\times_\rho\mathsf{V}$ the homogeneous holomorphic vector bundle over the complex flag manifold $G_{\mathbb{C}}/Q^-$ associated with ρ, and by $\iota^\sharp(G_{\mathbb{C}}\times_\rho\mathsf{V})$ its restriction to the domain $G/L=\iota(G/L)\subset G_{\mathbb{C}}/Q^-$.

$$\begin{array}{ccc}\iota^\sharp(G_{\mathbb{C}}\times_\rho\mathsf{V}) & & G_{\mathbb{C}}\times_\rho\mathsf{V}\\ \downarrow & & \downarrow\\ G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^-\end{array}$$

Then we consider

$$\begin{aligned}\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho) &:= \left\{h : G_{\mathbb{C}}\to\mathsf{V} \,\middle|\, \begin{array}{l}(1)\ h \text{ is holomorphic,}\\ (2)\ h(aq)=\rho(q)^{-1}\big(h(a)\big)\\ \qquad\text{for all } (a,q)\in G_{\mathbb{C}}\times Q^-\end{array}\right\}\ \text{and}\\ \mathcal{V}_{G/L}(\mathsf{V},\rho) &:= \left\{\psi : GQ^-\to\mathsf{V} \,\middle|\, \begin{array}{l}(1)\ \psi \text{ is holomorphic,}\\ (2)\ \psi(xq)=\rho(q)^{-1}\big(\psi(x)\big)\\ \qquad\text{for all } (x,q)\in GQ^-\times Q^-\end{array}\right\}\end{aligned}\tag{2.7}$$

as the complex vector spaces of holomorphic cross-sections of the bundles $G_{\mathbb{C}}\times_\rho\mathsf{V}$ and $\iota^\sharp(G_{\mathbb{C}}\times_\rho\mathsf{V})$, respectively, and sometimes express them as $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and $\mathcal{V}_{G/L}$, respectively. Furthermore, we define a homomorphism $\varrho : G\to GL(\mathcal{V}_{G/L})$, $g\mapsto\varrho(g)$, by

$$\big(\varrho(g)\psi\big)(x) := \psi(g^{-1}x) \text{ for } \psi\in\mathcal{V}_{G/L} \text{ and } x\in GQ^-,\tag{2.8}$$

and equip the vector space $\mathcal{V}_{G/L}$ with the Fréchet metric d. Here we refer to (2.6.3) in [3, p.230] for the definition of the metric d. Similar to the above, we define a homomorphism $\tilde{\varrho} : G_{\mathbb{C}}\to GL(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$, $a\mapsto\tilde{\varrho}(a)$, and provide the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ with a metric topology.

Remark 2.9. Here are comments on $(\mathcal{V}_{G/L},\varrho)$ and $(\mathcal{V}_{G_{\mathbb{C}}/Q^-},\tilde{\varrho})$.

(i) Lemma 2.6.18 in [3, p.235] shows that $\mathcal{F}:\mathcal{V}_{G_{\mathbb{C}}/Q^-}\to\mathcal{V}_{G/L}$, $h\mapsto h|_{GQ^-}$, is an injective linear mapping satisfying $\mathcal{F}\circ\tilde{\varrho}(g)=\varrho(g)\circ\mathcal{F}$ for all $g\in G$.
(ii) Lemma 2.5-(vii) implies that $G_{\mathbb{C}}/Q^-$ is a connected compact complex manifold, so that $\dim_{\mathbb{C}}\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho)<\infty$ for every finite-dimensional complex vector space V and holomorphic homomorphism $\rho:Q^-\to GL(\mathsf{V})$.[5]

In addition, given a $\psi\in\mathcal{V}_{G/L}$, we define a $\varrho(K)$-invariant complex vector subspace $\mathcal{U}_\psi\subset\mathcal{V}_{G/L}$ by

$$\mathcal{U}_\psi:=\operatorname{span}_{\mathbb{C}}\{\varrho(k)\psi\,|\,k\in K\},\tag{2.10}$$

and denote by $(\mathcal{V}_{G/L})_K$ the set of K-finite vectors in $\mathcal{V}_{G/L}$ for ϱ, i.e.,

$$(\mathcal{V}_{G/L})_K:=\{\phi\in\mathcal{V}_{G/L}\,|\,\dim_{\mathbb{C}}\mathcal{U}_\phi<\infty\}.\tag{2.11}$$

Note here that
(1) $\mathcal{F}(\mathcal{V}_{G_{\mathbb{C}}/Q^-})\subset(\mathcal{V}_{G/L})_K$,
(2) $\mathcal{V}_{G/L}=(\mathcal{V}_{G/L},d)$ is a complex Fréchet space, and ϱ is a continuous representation of the Lie group G on the Fréchet space $\mathcal{V}_{G/L}$,[6]
(3) the differential representation $\varrho_*:\mathfrak{g}_{\mathbb{C}}\to\mathfrak{gl}(\mathcal{V}_{G/L})$, $A\mapsto\varrho_*(A)$, of $\varrho:G\to GL(\mathcal{V}_{G/L})$ is well-defined,[7]
(4) $(\mathcal{V}_{G/L})_K$ is a $\varrho(K)$-invariant complex vector subspace of $\mathcal{V}_{G/L}$; besides, it is dense in $\mathcal{V}_{G/L}=(\mathcal{V}_{G/L},d)$,[8]
(5) $\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K<\infty$ implies $\dim_{\mathbb{C}}\mathcal{V}_{G/L}=\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K<\infty$.

Let us explain the reason why the above (5) holds. Suppose that $\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K<\infty$. Then $(\mathcal{V}_{G/L})_K$ is a closed subset of $\mathcal{V}_{G/L}=(\mathcal{V}_{G/L},d)$, since $\mathcal{V}_{G/L}$ is a Hausdorff topological vector space. Therefore one has $\mathcal{V}_{G/L}=(\mathcal{V}_{G/L})_K$, since $(\mathcal{V}_{G/L})_K$ is dense in $\mathcal{V}_{G/L}$. In view of $\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K<\infty$ and $\mathcal{V}_{G/L}=(\mathcal{V}_{G/L})_K$ we see that $\dim_{\mathbb{C}}\mathcal{V}_{G/L}=\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K<\infty$.

We end this subsection with defining a complex vector subspace $\mathcal{W}_{G/L}\subset\mathcal{V}_{G/L}$ by

$$\mathcal{W}_{G/L}:=\{\psi\in\mathcal{V}_{G/L}\,|\,\psi(u)=\psi(e)\text{ for all }u\in(U^+\cap GQ^-)_e\},\tag{2.12}$$

where $(U^+\cap GQ^-)_e$ denotes the connected component of $U^+\cap GQ^-$ containing the unit element $e\in G_{\mathbb{C}}$.

2.4. **Four lemmas.** In the setting of §§2.2, §§2.3 we state the following four Lemmas 2.13, 2.14, 2.15 and 2.16 which are needed later:

Lemma 2.13. *$\varrho_*(A)\phi\in(\mathcal{V}_{G/L})_K$ for all $(A,\phi)\in\mathfrak{g}_{\mathbb{C}}\times(\mathcal{V}_{G/L})_K$.*

Proof. For any $(A,\phi)\in\mathfrak{g}_{\mathbb{C}}\times(\mathcal{V}_{G/L})_K$, we are going to confirm that the vector space $\mathcal{U}_{\varrho_*(A)\phi}=\operatorname{span}_{\mathbb{C}}\{\varrho(k)(\varrho_*(A)\phi)\,|\,k\in K\}$ is finite-dimensional. Let $\mathcal{R}$ be a complex

[5] cf. Corollary in Kodaira [10, p.161].
[6] e.g. the proof of Lemma 2.6.4 in [3, p.230] and references therein.
[7] cf. [3, §§2.6.2], $\big(\varrho_*(A)\psi\big)(x)=d/dt|_{t=0}\psi\big(\exp(-tA)x\big)$ for $\psi\in\mathcal{V}_{G/L}$ and $x\in GQ^-$.
[8] e.g. Proposition 6.2.1 in [2, p.62].

vector subspace of $\mathcal{V}_{G/L}$ defined by $\mathcal{R} := \mathrm{span}_{\mathbb{C}}\{\varrho_*(B)\varphi \,|\, B \in \mathfrak{g}_{\mathbb{C}}, \varphi \in \mathcal{U}_\phi\}$. Then, it comes from $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} < \infty$ and $\dim_{\mathbb{C}} \mathcal{U}_\phi < \infty$ that

$$\dim_{\mathbb{C}} \mathcal{R} < \infty.$$

Now, let us show that $\mathcal{R}$ is $\varrho(K)$-invariant. For each $k \in K$ and $(B, \varphi) \in \mathfrak{g}_{\mathbb{C}} \times \mathcal{U}_\phi$, one has $\mathrm{Ad}\, k(B) \in \mathfrak{g}_{\mathbb{C}}$ and $\varrho(k)\varphi \in \mathcal{U}_\phi$ (since $\mathcal{U}_\phi$ is $\varrho(K)$-invariant), and moreover

$$\mathcal{R} \ni \varrho_*(\mathrm{Ad}\, k(B))(\varrho(k)\varphi) \stackrel{(2.8)}{=} \varrho(k)(\varrho_*(B)\varphi).$$

This shows that $\mathcal{R}$ is $\varrho(K)$-invariant. Accordingly $\varrho_*(A)\phi \in \mathcal{R}$ yields $\mathcal{U}_{\varrho_*(A)\phi} \subset \mathcal{R}$, and $\dim_{\mathbb{C}} \mathcal{U}_{\varrho_*(A)\phi} \le \dim_{\mathbb{C}} \mathcal{R} < \infty$. □

Lemma 2.14. *For any holomorphic function $h : GQ^- \to \mathbb{C}$ and non-empty open subset O^+ of $U^+ \cap GQ^-$, the restriction $h|_{O^+}$ is holomorphic on O^+.*

Proof. From Lemma 2.5-(iii), (ix) it follows that the inclusion $\imath : U^+ \to G_{\mathbb{C}}$, $u \mapsto u$, is holomorphic, and that $U^+ \cap GQ^-$ and GQ^- are open subsets of U^+ and $G_{\mathbb{C}}$, respectively. These imply that $\imath : U^+ \cap GQ^- \to GQ^-$ is holomorphic; furthermore, $\imath : O^+ \to GQ^-$ is holomorphic. Therefore $h \circ \imath = h|_{O^+}$ is holomorphic. □

Lemma 2.15. *Suppose that*

(s1) *the representation $\rho : Q^- \to GL(\mathsf{V})$, $q \mapsto \rho(q)$, is irreducible,*

and that $\mathcal{V}' \neq \{0\}$ for a given closed $\varrho(G)$-invariant complex vector subspace $\mathcal{V}'$ of $\mathcal{V}_{G/L} = (\mathcal{V}_{G/L}, d)$. Then, it turns out that $\{0\} \neq \mathcal{W}_{G/L} \subset \mathcal{V}'$.

Proof. We refer to the proof of Proposition 3.2.1-(1) in [3, p.239]. □

Lemma 2.16. *Suppose that*

(s1) *$\rho : Q^- \to GL(\mathsf{V})$ is irreducible, and*

(I) *$\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$.*

Then, the following two items hold:

(II) *ϱ is an irreducible representation of G on $\mathcal{V}_{G/L}$.*

(III) *The representation module $(\mathcal{V}_{G_{\mathbb{C}}/Q^-}, \tilde{\varrho})$ is G-equivariant isomorphic to $(\mathcal{V}_{G/L}, \varrho)$ via $\mathcal{F} : h \mapsto h|_{GQ^-}$.*

Proof. If $\mathcal{V}_{G/L} = \{0\}$, then Remark 2.9-(i) tells us that $\mathcal{V}_{G_{\mathbb{C}}/Q^-} = \{0\}$, so that both (II) and (III) hold. Hereafter, we investigate the case $\mathcal{V}_{G/L} \neq \{0\}$.

(II) First, let us confirm (II). Let $\mathcal{V}_1$ be any closed $\varrho(G)$-invariant complex vector subspace of $\mathcal{V}_{G/L} = (\mathcal{V}_{G/L}, d)$ with $\mathcal{V}_1 \neq \{0\}$. Then, the supposition (I) and Weyl's theorem on semisimplicity of representations[9] enable us to choose a closed $\varrho(G)$-invariant complex vector subspace $\mathcal{V}_2$ of $\mathcal{V}_{G/L}$ so that

$$\mathcal{V}_{G/L} = \mathcal{V}_1 \oplus \mathcal{V}_2$$

because the Lie algebra $\mathfrak{g}$ is semisimple and the Lie group G is connected. If $\mathcal{V}_2 \neq \{0\}$, then the supposition (s1) and Lemma 2.15 yield $\{0\} \neq \mathcal{W}_{G/L} \subset \mathcal{V}_1 \cap \mathcal{V}_2$, which is a contradiction. Therefore $\mathcal{V}_2 = \{0\}$, and $\mathcal{V}_1 = \mathcal{V}_{G/L}$. Hence (II) holds.

[9]cf. Theorem 3.13.1 in Varadarajan [15, p.222].

(III) It suffices to confirm that

ⓐ $$\mathcal{V}_{G_{\mathbb{C}}/Q^-} \neq \{0\}$$

by virtue of (II) and Remark 2.9-(i). Let Π_1 be a fundamental root system of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ satisfying (2.4), and $\triangle^+$ the set of positive roots in $\triangle$ relative to Π_1. From $\mathcal{V}_{G/L} \neq \{0\}$, (I) and (II) there exist a non-zero $\varphi \in \mathcal{V}_{G/L}$ and a unique linear function $\nu : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ such that

$$\text{(2.i) } \varrho_*(\mathfrak{g}_\alpha)\varphi = \{0\} \text{ for all } \alpha \in \triangle^+, \quad \text{(2.ii) } \varrho_*(A)\varphi = \nu(A)\varphi \text{ for all } A \in \mathfrak{h}_{\mathbb{C}}.$$

Here ν is a dominant integral form on $\mathfrak{h}_{\mathbb{C}}$ with respect to $\triangle^+$, and it follows from (2.i) and $\triangle(T, +) \subset \triangle^+$ that $\varphi \in \mathcal{W}_{G/L}$, so that

$$\varphi(e) \neq 0$$

due to $\varphi \neq 0$ and Lemma 2.6.17 in [3, p.234]. Accordingly one can prove the above ⓐ by holomorphic induction in stages, while referring to the proof of the item ⓐ in [3, p.244]. □

2.5. **An analytic continuation from G/L to $G_{\mathbb{C}}/Q^-$.** The setting of §§2.2, §§2.3 remains valid here. This subsection outlines a proof of Theorem 2.17 below and recalls two lemmas. Here, this theorem was originally proved in [1] and its proof is improved in [2].

Theorem 2.17 (cf. Theorem 3.1 in [1, p.10]). *Suppose that there exists a fundamental root system Π_1 of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ satisfying* (2.4) *and*

(s3) $\mathfrak{g}_\beta \subset \mathfrak{k}_{\mathbb{C}}$ *for every* $\beta \in \Pi_1 - \triangle(T, 0)$.

Then, the complex vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ is linear isomorphic to $\mathcal{V}_{G/L}$ via

$$\mathcal{F} : \mathcal{V}_{G_{\mathbb{C}}/Q^-} \to \mathcal{V}_{G/L}, \; h \mapsto h|_{GQ^-},$$

and so $\dim_{\mathbb{C}} \mathcal{V}_{G/L} = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$. *Here $h|_{GQ^-}$ is the restriction of h to GQ^-.*

We will outline a proof of this theorem (see [2, §§8.3, §§11.2.2] for related matters). Define a Weyl group $\mathfrak{W}$ of the complex semisimple Lie group $G_{\mathbb{C}}$, an action ζ of $\mathfrak{W}$ on the dual space $(\mathfrak{h}_{\mathbb{C}})^*$, and an element $w_\alpha \in N_{G_u}(i\mathfrak{h}_{\mathbb{R}})$ by

$$\mathfrak{W} := N_{G_u}(i\mathfrak{h}_{\mathbb{R}})/C_{G_u}(i\mathfrak{h}_{\mathbb{R}}), \quad \zeta([w])\eta := {}^t\mathrm{Ad}\, w^{-1}(\eta) \text{ for } [w] \in \mathfrak{W} \text{ and } \eta \in (\mathfrak{h}_{\mathbb{C}})^*,$$
$$w_\alpha := \exp(\pi/2)(E_\alpha - E_{-\alpha}) \text{ for an } \alpha \in \triangle,$$

respectively, where $[w]$ stands for the left coset $wC_{G_u}(i\mathfrak{h}_{\mathbb{R}})$. Let $\Pi_1 \subset \triangle$ be a fundamental root system satisfying (2.4). Relative to this Π_1 we fix the set $\triangle^+$ (resp. $\triangle^-$) of positive (resp. negative) roots in $\triangle$, and define a simply connected, closed complex nilpotent subgroup N^+ of $G_{\mathbb{C}}$ by $N^+ := \exp \bigoplus_{\alpha \in \triangle^+} \mathfrak{g}_\alpha$. In addition, we set

$$\Phi_{[w]} := \{\alpha \in \triangle^+ \,|\, \zeta([w])^{-1}\alpha \in \triangle^-\} \text{ for a } [w] \in \mathfrak{W},$$
$$\mathfrak{W}^1 := \{[w] \in \mathfrak{W} \,|\, \Phi_{[w]} \subset \triangle^+ - \triangle(T, 0)\}.$$

Then, N^+ acts on the complex flag manifold $G_{\mathbb{C}}/Q^-$ via $N^+ \times G_{\mathbb{C}}/Q^- \ni (n, aQ^-) \mapsto naQ^- \in G_{\mathbb{C}}/Q^-$, and $\mathfrak{W}^1$ forms all representatives for the orbit space $(G_{\mathbb{C}}/Q^-)/N^+$, i.e., $G_{\mathbb{C}}/Q^- = \bigcup_{[\sigma]\in\mathfrak{W}^1} N^+\sigma^{-1}Q^-/Q^-$ (disjoint union). This gives rise to a generalized Bruhat decomposition

$$G_{\mathbb{C}} = \bigcup_{[\sigma]\in\mathfrak{W}^1} N^+\sigma^{-1}Q^- \text{ (disjoint union)}$$

and one knows that for a given $[\sigma]\in\mathfrak{W}^1$,

- $\dim_{\mathbb{C}} N^+\sigma^{-1}Q^- = \dim_{\mathbb{C}} G_{\mathbb{C}}$ if and only if $[\sigma]=[e]$,
- $\dim_{\mathbb{C}} N^+\sigma^{-1}Q^- = \dim_{\mathbb{C}} G_{\mathbb{C}}-1$ if and only if there exists a $\beta\in\Pi_1-\triangle(T,0)$ satisfying $[\sigma]=[w_\beta]$,

where e is the unit element of $G_{\mathbb{C}}$. cf. Kostant [11], [12]. Setting

$$O_1 := N^+Q^- \cup \bigcup_{\beta\in\Pi_1-\triangle(T,0)} N^+w_\beta^{-1}Q^-,$$

we conclude that every holomorphic function f on O_1 can be continued analytically to the whole $G_{\mathbb{C}}$ by Hartogs's continuation theorem (since $G_{\mathbb{C}}-O_1$ is of codimension 2 or more).

Now, let us outline a proof of Theorem 2.17. Remark that

(C) for each $\phi\in(\mathcal{V}_{G/L})_K$ there exists a unique holomorphic mapping $\phi' : U^+\to \mathsf{V}$ such that $\phi=\phi'$ on $(U^+\cap GQ^-)_e$,

where $(U^+\cap GQ^-)_e$ is the connected component of $U^+\cap GQ^-$ containing the unit $e\in G_{\mathbb{C}}$. On the one hand, it turns out that

$$N^+Q^- = U^+Q^-,\quad N^+w_\beta^{-1}Q^- \subset w_\beta^{-1}U^+Q^- \text{ for all } \beta\in\Pi_1-\triangle(T,0)$$

under the supposition (2.4). Thus, every holomorphic function f on $O_2 := U^+Q^-\cup \bigcup_{\beta\in\Pi_1-\triangle(T,0)} w_\beta^{-1}U^+Q^-$ can be continued analytically to the whole $G_{\mathbb{C}}$. On the other hand, the supposition (s3) in Theorem 2.17 implies that w_β belongs to K for every $\beta\in\Pi_1-\triangle(T,0)$. Accordingly, for any $\phi\in(\mathcal{V}_{G/L})_K$ and all $\beta\in\Pi_1-\triangle(T,0)$ we have $\varrho(w_\beta)\phi\in(\mathcal{V}_{G/L})_K$, and it follows from (C) and (2.7)-(2) that all the restrictions of ϕ and $\varrho(w_\beta)\phi$ to $(U^+\cap GQ^-)_e$ can be continued analytically to U^+Q^-. From that, ϕ can be continued analytically to the O_2; moreover, to the $G_{\mathbb{C}}$. This assures that for any $\phi\in(\mathcal{V}_{G/L})_K$ there exists a unique $h\in\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ satisfying $\phi=h|_{GQ^-}$, so that $(\mathcal{V}_{G/L})_K=\mathcal{F}(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$, $\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K=\dim_{\mathbb{C}}\mathcal{V}_{G_{\mathbb{C}}/Q^-}<\infty$. Therefore Theorem 2.17 holds (see (5) on page 8 and Remark 2.9-(i)). Incidentally, the above (C) comes from the following two lemmas which are needed later:

Lemma 2.18 (cf. Lemma 11.2.12 in [2, p.122])**.** *For any $\vartheta\in\mathbb{R}$ and $b_1,b_2,\dots,b_N>0$, the number of non-negative integer solutions $(p_1,p_2,\dots,p_N)$ to the equation*

$$\vartheta = b_1p_1+b_2p_2+\cdots+b_Np_N$$

is only finite or zero.

Lemma 2.19 (cf. Lemma 11.2.14 in [2, p.122])**.** (1) *Given a $\phi\in(\mathcal{V}_{G/L})_K-\{0\}$, there exist a complex basis $\{\phi_a\}_{a=1}^{k_\phi}$ of $\mathcal{U}_\phi$ and $\mu_1,\mu_2,\dots,\mu_{k_\phi}\in\mathbb{R}$ such that*

$$\varrho(\exp tT)\phi_a = e^{i\mu_a t}\phi_a$$

for all $1\le a\le k_\phi=\dim_{\mathbb{C}}\mathcal{U}_\phi$ and $t\in\mathbb{R}$.

(2) *If* $\mathsf{V} \neq \{0\}$, *then there exist a complex basis* $\{\mathsf{v}_b\}_{b=1}^m$ *of* V *and* $\theta_1, \theta_2, \dots, \theta_m \in \mathbb{R}$ *such that*

$$\rho(\exp tT)\mathsf{v}_b = e^{i\theta_b t}\mathsf{v}_b$$

for all $1 \leq b \leq m = \dim_{\mathbb{C}} \mathsf{V}$ *and* $t \in \mathbb{R}$.

3. Proof of Theorem 1.2

In this section also, we obey the setting of §§2.2 and §§2.3.

This section consists of three subsections. In §§3.1 we set a special subset $\Lambda \subset \triangle$. In §§3.2 we deal with the case where $\mathfrak{g}$ has a compact Cartan subalgebra and derive $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$ from the supposition $\triangle \subset \Lambda$. In §§3.3 we deal with the case where $\mathfrak{g}$ is a simple Lie algebra of Hermitian type, introduce a fundamental root system Π_2 by taking the structure of irreducible Hermitian symmetric space of non-compact type into consideration, and establish two Corollaries 3.14 and 3.15 and Proposition 3.17 which enable us to complete the proof of Theorem 1.2.

3.1. **A special subset $\Lambda \subset \triangle$.** Let Λ denote the set of roots $\alpha \in \triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which satisfy the following condition (3.1):

(3.1) for each $\phi \in (\mathcal{V}_{G/L})_K$ there exists an $n \in \mathbb{N}$ such that $\varrho_*(E_\alpha)^n\phi = 0$,

where we express $\underbrace{\varrho_*(E_\alpha)(\cdots(\varrho_*(E_\alpha)(\varrho_*(E_\alpha)}_{n}\phi))\cdots)$ as $\varrho_*(E_\alpha)^n\phi$. With this notation Λ, we demonstrate

Proposition 3.2. $\beta \in \Lambda$ *for all* $\beta \in \triangle(T,+)$, *in other words,* $\mathfrak{g}_\beta \subset \mathfrak{u}^+$, $\beta \in \triangle$ *imply* $\beta \in \Lambda$.

Proof. Take any $\beta \in \triangle(T,+)$ and $\phi \in (\mathcal{V}_{G/L})_K$. Let us confirm that there exists an $n \in \mathbb{N}$ satisfying

$$\varrho_*(E_\beta)^n\phi = 0.$$

That is evident when $\phi = 0$. Hereafter, we investigate the case $\phi \neq 0$. In this case, there exist a complex basis $\{\phi_a\}_{a=1}^{k_\phi}$ of $\mathcal{U}_\phi$ and $\mu_1, \mu_2, \dots, \mu_{k_\phi} \in \mathbb{R}$ such that

① $$\varrho(\exp tT)\phi_a = e^{i\mu_a t}\phi_a$$

for all $1 \leq a \leq k_\phi = \dim_{\mathbb{C}} \mathcal{U}_\phi$ and $t \in \mathbb{R}$, and there exist a complex basis $\{\mathsf{v}_b\}_{b=1}^m$ of V and $\theta_1, \theta_2, \dots, \theta_m \in \mathbb{R}$ such that

② $$\rho(\exp tT)\mathsf{v}_b = e^{i\theta_b t}\mathsf{v}_b$$

for all $1 \leq b \leq m = \dim_{\mathbb{C}} \mathsf{V}$ and $t \in \mathbb{R}$, by Lemma 2.19.

First, let us fix the setting for this proof. We recall $\triangle(T,+) = \{\beta_\ell\}_{\ell=1}^N$, assume

Ⓐ $$\beta = \beta_1$$

by changing the indexes of $\beta_1, \beta_2, \dots, \beta_N$ (if necessary), and express $\phi_a(x) \in \mathsf{V}$ as

③ $$\phi_a(x) = {\phi_a}^1(x)\mathsf{v}_1 + {\phi_a}^2(x)\mathsf{v}_2 + \cdots + {\phi_a}^m(x)\mathsf{v}_m \quad (1 \leq a \leq k_\phi,\ x \in GQ^-).$$

Let $(w^1, w^2, \dots, w^N)$ denote the canonical coordinates of the second kind associated with the basis $\{E_{\beta_\ell}\}_{\ell=1}^N \subset \mathfrak{u}^+$,

$$U^+ \supset O^+ \ni \exp(w^1 E_{\beta_1}) \exp(w^2 E_{\beta_2}) \cdots \exp(w^N E_{\beta_N}) \longleftrightarrow (w^1, w^2, \dots, w^N) \in O \subset \mathbb{C}^N.$$

Here O^+ and O are some open subsets of U^+ and $\mathbb{C}^N$ such that

$$e \in O^+ \subset U^+ \cap GQ^-$$

and $(0, 0, \dots, 0) \in O$, respectively. For each $1 \leq a \leq k_\phi$ and $1 \leq b \leq m$, the function ${\phi_a}^b : GQ^- \to \mathbb{C}$ is holomorphic; thus the restriction ${\phi_a}^b|_{O^+}$ is also holomorphic due to Lemma 2.14. Accordingly, there exists an $R > 0$ so that the following conditions (a1) and (a2) hold for $P := \{u \in O^+ : |w^\ell(u)| < R, 1 \leq \ell \leq N\}$:

(a1) P is an open subset of O^+ containing e,

(a2) on P we can express each ${\phi_a}^b|_{O^+}$ as

$${\phi_a}^b(w^1, w^2, \dots, w^N) = \sum_{p_1, p_2, \dots, p_N \geq 0} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} (w^1)^{p_1} (w^2)^{p_2} \cdots (w^N)^{p_N}$$

(the Taylor expansion of ${\phi_a}^b|_{O^+}$ at $e = (0, 0, \dots, 0)$).

In this setting, we will prove that there exists an $n \in \mathbb{N}$ satisfying $\varrho_*(E_\beta)^n \phi = 0$.

Our first aim is to demonstrate that for each $1 \leq a \leq k_\phi$ and $1 \leq b \leq m$,

ⓐ ${\phi_a}^b(w^1, w^2, \dots, w^N) = \sum_{p_1, p_2, \dots, p_N \geq 0} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} (w^1)^{p_1} (w^2)^{p_2} \cdots (w^N)^{p_N}$ is a polynomial function on the P of finite degree.

For any $u = \exp(w^1 E_{\beta_1}) \exp(w^2 E_{\beta_2}) \cdots \exp(w^N E_{\beta_N}) \in P$ and sufficiently small $t \in \mathbb{R}$, we see from $\beta_\ell(T) \in i\mathbb{R}$, $(\exp tT) u \exp(-tT) \in O^+$ that $|e^{\beta_\ell(T)t} w^\ell| = |w^\ell| < R$,

$$\begin{aligned}&(\exp tT) u \exp(-tT) \\ &\qquad = \exp(e^{\beta_1(T)t} w^1 E_{\beta_1}) \exp(e^{\beta_2(T)t} w^2 E_{\beta_2}) \cdots \exp(e^{\beta_N(T)t} w^N E_{\beta_N}) \in P.\end{aligned}$$

This and (a2) assure that for each $1 \leq a \leq k_\phi$ and $1 \leq b \leq m$,

④ $$\begin{aligned}&{\phi_a}^b\big((\exp tT) u \exp(-tT)\big) = {\phi_a}^b(e^{\beta_1(T)t} w^1, e^{\beta_2(T)t} w^2, \dots, e^{\beta_N(T)t} w^N) \\ &\quad = \sum_{p_1, p_2, \dots, p_N \geq 0} e^{(\beta_1(T)p_1 + \beta_2(T)p_2 + \cdots + \beta_N(T)p_N)t} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} (w^1)^{p_1} (w^2)^{p_2} \cdots (w^N)^{p_N}.\end{aligned}$$

For any $1 \leq a \leq k_\phi$ we obtain

$$\begin{aligned}\sum_{b=1}^m e^{i\theta_b t} {\phi_a}^b(u) \mathsf{v}_b &\overset{②}{=} \rho(\exp tT)\Big(\sum_{b=1}^m {\phi_a}^b(u) \mathsf{v}_b\Big) \\ &\overset{③}{=} \rho(\exp tT)\big(\phi_a(u)\big) \\ &= \phi_a(u \exp(-tT)) \qquad (\because \phi_a \in \mathcal{V}_{G/L},\ (2.7)\text{-}(2)) \\ &\overset{(2.8)}{=} (\varrho(\exp tT)\phi_a)\big((\exp tT) u \exp(-tT)\big) \\ &\overset{①}{=} e^{i\mu_a t} \phi_a\big((\exp tT) u \exp(-tT)\big)\end{aligned}$$

$$\overset{③}{=} \sum_{b=1}^{m} e^{i\mu_a t} {\phi_a}^b\big((\exp tT) u \exp(-tT)\big) \mathsf{v}_b.$$

This and (a2) assure that for each $1 \leq a \leq k_\phi$ and $1 \leq b \leq m$,

$$⑤ \quad {\phi_a}^b\big((\exp tT) u \exp(-tT)\big) = e^{i(\theta_b - \mu_a)t} {\phi_a}^b(u) = e^{i(\theta_b-\mu_a)t} {\phi_a}^b(w^1, w^2, \dots, w^N)$$
$$= \sum_{p_1, p_2, \dots, p_N \geq 0} e^{i(\theta_b-\mu_a)t} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} (w^1)^{p_1} (w^2)^{p_2} \cdots (w^N)^{p_N}.$$

These ④ and ⑤ yield

$$e^{i(\theta_b-\mu_a)t} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} = e^{i(\beta_1(-iT)p_1 + \beta_2(-iT)p_2 + \cdots + \beta_N(-iT)p_N)t} {\alpha_a}^b{}_{p_1 p_2 \cdots p_N}$$

for all $1 \leq a \leq k_\phi$, $1 \leq b \leq m$, $0 \leq p_1, p_2, \dots, p_N \in \mathbb{Z}$ and sufficiently small $t \in \mathbb{R}$. Differentiating this equation at $t = 0$ we have

$$(\theta_b - \mu_a) {\alpha_a}^b{}_{p_1 p_2 \cdots p_N} = \big(\beta_1(-iT)p_1 + \beta_2(-iT)p_2 + \cdots + \beta_N(-iT)p_N\big) {\alpha_a}^b{}_{p_1 p_2 \cdots p_N}.$$

Here Lemma 2.18 and $\beta_\ell(-iT) > 0$ imply that for each $1 \leq a \leq k_\phi$ and $1 \leq b \leq m$, the number of non-negative integer solutions $(p_1, p_2, \dots, p_N)$ to the equation

$$\theta_b - \mu_a = \beta_1(-iT)p_1 + \beta_2(-iT)p_2 + \cdots + \beta_N(-iT)p_N$$

is only finite or zero, so that the number of the non-zero coefficients ${\alpha_a}^b{}_{p_1 p_2 \cdots p_N}$ is only finite. Hence ⓐ holds.

By virtue of ⓐ, one can choose an $n \in \mathbb{N}$ so that

$$⑥ \qquad \frac{\partial^n {\phi_a}^b}{\partial (w^1)^n} = 0 \text{ on } P \quad (1 \leq a \leq k_\phi,\ 1 \leq b \leq m).$$

For any $1 \leq a \leq k_\phi$ and $p = \exp(\alpha^1 E_{\beta_1}) \exp(\alpha^2 E_{\beta_2}) \cdots \exp(\alpha^N E_{\beta_N}) \in P$, one has

$$\begin{aligned}
\big(\varrho_*(E_\beta)\phi_a\big)(p) &= \sum_{b=1}^{m} \frac{d}{dt}\Big|_{t=0} {\phi_a}^b\big(\exp(-tE_\beta)p\big)\mathsf{v}_b \\
&\overset{Ⓐ}{=} \sum_{b=1}^{m} \frac{d}{dt}\Big|_{t=0} {\phi_a}^b\big(\exp((-t+\alpha^1)E_{\beta_1}) \exp(\alpha^2 E_{\beta_2}) \cdots \exp(\alpha^N E_{\beta_N})\big)\mathsf{v}_b \\
&= \sum_{b=1}^{m} \frac{d}{dt}\Big|_{t=0} {\phi_a}^b\big(-t+\alpha^1, \alpha^2, \dots, \alpha^N\big)\mathsf{v}_b \\
&= -\sum_{b=1}^{m} \frac{\partial {\phi_a}^b}{\partial w^1}(p)\mathsf{v}_b,
\end{aligned}$$

and furthermore,

$$\big(\varrho_*(E_\beta)^n \phi_a\big)(p) = (-1)^n \sum_{b=1}^{m} \frac{\partial^n {\phi_a}^b}{\partial (w^1)^n}(p)\mathsf{v}_b.$$

Consequently, it is immediate from ⑥ that $\varrho_*(E_\beta)^n \phi_a = 0$ on P $(1 \leq a \leq k_\phi)$, so that

$$\varrho_*(E_\beta)^n \phi = 0 \text{ on } P$$

because of $\phi \in \mathcal{U}_\phi = \mathrm{span}_{\mathbb{C}}\{\phi_a\}_{a=1}^{k_\phi}$. This and Lemma 2.6.15-(4) in [3, p.234] enable us to conclude that $\varrho_*(E_\beta)^n\phi = 0$ on the whole GQ^-. □

3.2. **Case $\mathfrak{g}$ has a compact Cartan subalgebra.** Suppose that the Lie algebra $\mathfrak{g}$ has a compact Cartan subalgebra. Then it turns out that $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$ and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \textstyle\bigoplus_{\alpha\in\triangle} \mathfrak{g}_\alpha;$$

besides, either $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}_{\mathbb{C}}$ holds for each $\alpha \in \triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. A root $\alpha \in \triangle$ is said to be *compact* (resp. *non-compact*) if $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$ (resp. $\mathfrak{g}_\alpha \subset \mathfrak{p}_{\mathbb{C}}$). We denote by $\triangle(\mathfrak{k})$ and $\triangle(\mathfrak{p})$ the sets of compact and non-compact roots in $\triangle$, respectively, and have

$$\begin{aligned} &\triangle = \triangle(\mathfrak{k}) \cup \triangle(\mathfrak{p}) \text{ (disjoint union)},\\ &\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \textstyle\bigoplus_{\alpha\in\triangle(\mathfrak{k})} \mathfrak{g}_\alpha, \quad \mathfrak{p}_{\mathbb{C}} = \bigoplus_{\alpha\in\triangle(\mathfrak{p})} \mathfrak{g}_\alpha. \end{aligned} \tag{3.3}$$

With the notation Λ in §§3.1 and this notation $\triangle(\mathfrak{k})$, we assert

Lemma 3.4. *Suppose that* (s4) $\mathfrak{g}$ *has a compact Cartan subalgebra. Then, $\alpha \in \Lambda$ for all $\alpha \in \triangle(\mathfrak{k})$, in other words, $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$, $\alpha \in \triangle$ imply $\alpha \in \Lambda$.*

Proof. Take any $\alpha \in \triangle(\mathfrak{k})$ and $\phi \in (\mathcal{V}_{G/L})_K$. Since $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$, $\dim_{\mathbb{C}} \mathcal{U}_\phi < \infty$ and $\mathcal{U}_\phi$ is $\varrho(K)$-invariant, there exists an $n \in \mathbb{N}$ such that $\varrho_*(E_\alpha)^n\phi = 0$. Thus $\alpha \in \Lambda$ by (3.1). □

Let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then the representation ϱ_* of $\mathfrak{g}_{\mathbb{C}}$ on $\mathcal{V}_{G/L}$ can be extended to a unique representation of $U(\mathfrak{g}_{\mathbb{C}})$. We denote this extension by the same notation ϱ_*, and express $\varrho_*({E_1}^{i_1}{E_2}^{i_2}\cdots{E_n}^{i_n})\psi$ as

$$\varrho_*(E_1)^{i_1}\varrho_*(E_2)^{i_2}\cdots\varrho_*(E_n)^{i_n}\psi$$

for ${E_1}^{i_1}{E_2}^{i_2}\cdots{E_n}^{i_n} \in U(\mathfrak{g}_{\mathbb{C}})$ and $\psi \in \mathcal{V}_{G/L}$. In this setting we prove

Proposition 3.5. *Suppose that*

(s4) $\mathfrak{g}$ *has a compact Cartan subalgebra, and*
(s5) $\triangle \subset \Lambda$.

Then, the following (1), (2) *and* (3) *hold for a given non-zero $\varphi \in (\mathcal{V}_{G/L})_K$ endowed with a linear function $\omega : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ such that $\varrho_*(H)\varphi = \omega(H)\varphi$ for all $H \in \mathfrak{t}_{\mathbb{C}}$:*

(1) $0 < \dim_{\mathbb{C}} \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi < \infty$.
(2) *$\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ is a closed $\varrho(G)$-invariant complex vector subspace of $\mathcal{V}_{G/L}$ with respect to the Fréchet metric d.*
(3) $\mathcal{U}_\varphi \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$.

In addition; if (s1) *$\rho : Q^- \to GL(\mathsf{V})$ is irreducible, then the following* (4) *and* (5) *hold further:*

(4) $\{0\} \neq \mathcal{W}_{G/L} \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$.
(5) *ϱ is an irreducible representation of G on $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$.*

Proof. We suppose $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ to consist of r-roots $\alpha_r, \dots, \alpha_2, \alpha_1$,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\alpha_r} \oplus \cdots \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1}.$$

(1) We will only prove $\dim_{\mathbb{C}} \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi < \infty$. From (s5) we obtain $\alpha_1 \in \Lambda$, and there exists an $N_1 \in \mathbb{N}$ such that

$$\varrho_*(E_{\alpha_1})^{N_1}\varphi = 0$$

by (3.1) and $\varphi \in (\mathcal{V}_{G/L})_K$. For each $0 \leq j \leq N_1$, Lemma 2.13 yields $\varrho_*(E_{\alpha_1})^j\varphi \in (\mathcal{V}_{G/L})_K$. This, together with $\alpha_2 \in \Lambda$ and (3.1), enables us to find an $n_{j2} \in \mathbb{N}$ which satisfies $\varrho_*(E_{\alpha_2})^{n_{j2}}\varrho_*(E_{\alpha_1})^j\varphi = 0$. Setting $N_2 := \max\{n_{j2} \,|\, 0 \leq j \leq N_1\}$, we see that $N_2 \in \mathbb{N}$ and

$$\varrho_*(E_{\alpha_2})^{N_2}\varrho_*(E_{\alpha_1})^j\varphi = 0 \text{ for all } 0 \leq j \leq N_1.$$

For each $0 \leq k \leq N_2$ and $0 \leq j \leq N_1$, we obtain $\varrho_*(E_{\alpha_2})^k\varrho_*(E_{\alpha_1})^j\varphi \in (\mathcal{V}_{G/L})_K$, and there exists an $n_{kj3} \in \mathbb{N}$ which satisfies $\varrho_*(E_{\alpha_3})^{n_{kj3}}\varrho_*(E_{\alpha_2})^k\varrho_*(E_{\alpha_1})^j\varphi = 0$ by $\alpha_3 \in \Lambda$ and (3.1). Setting $N_3 := \max\{n_{kj3} \,|\, 0 \leq k \leq N_2, 0 \leq j \leq N_1\}$, we deduce that $N_3 \in \mathbb{N}$ and

$$\varrho_*(E_{\alpha_3})^{N_3}\varrho_*(E_{\alpha_2})^k\varrho_*(E_{\alpha_1})^j\varphi = 0 \text{ for all } 0 \leq k \leq N_2,\ 0 \leq j \leq N_1.$$

Inductively, we set natural numbers $N_4, N_5, \dots, N_r$. Then, one has

$$\text{①} \quad \begin{aligned} &\mathrm{span}_{\mathbb{C}}\{\varrho_*(E_{\alpha_r})^{m_r}\cdots\varrho_*(E_{\alpha_2})^{m_2}\varrho_*(E_{\alpha_1})^{m_1}\varphi \,|\, 0 \leq m_i \in \mathbb{Z}, 1 \leq i \leq r\} \\ &\quad = \mathrm{span}_{\mathbb{C}}\{\varrho_*(E_{\alpha_r})^{k_r}\cdots\varrho_*(E_{\alpha_2})^{k_2}\varrho_*(E_{\alpha_1})^{k_1}\varphi \,|\, 0 \leq k_i < N_i, 1 \leq i \leq r\}, \end{aligned}$$

and this vector space (on the right-hand side) is finite-dimensional. Here, since $\varrho_*(H)\varphi = \omega(H)\varphi$ for all $H \in \mathfrak{t}_{\mathbb{C}}$, it follows that

$$\text{②} \quad \begin{aligned} &\varrho_*(H)\varrho_*(E_{\alpha_r})^{m_r}\cdots\varrho_*(E_{\alpha_2})^{m_2}\varrho_*(E_{\alpha_1})^{m_1}\varphi \\ &= (\omega + m_r\alpha_r + \cdots + m_2\alpha_2 + m_1\alpha_1)(H)\varrho_*(E_{\alpha_r})^{m_r}\cdots\varrho_*(E_{\alpha_2})^{m_2}\varrho_*(E_{\alpha_1})^{m_1}\varphi, \\ &\qquad (\omega + m_r\alpha_r + \cdots + m_2\alpha_2 + m_1\alpha_1)(H) \in \mathbb{C}, \end{aligned}$$

for all $H \in \mathfrak{t}_{\mathbb{C}}$. Accordingly we conclude $\dim_{\mathbb{C}} \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi < \infty$ by ①, ② and the Poincaré-Birkhoff-Witt theorem.

(2) The vector space $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ is a closed subset of $\mathcal{V}_{G/L} = (\mathcal{V}_{G/L}, d)$ by virtue of (1), and it is also $\varrho(G)$-invariant, since $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ is $\varrho_*(\mathfrak{g})$-invariant and the Lie group G is connected.

(3) From (2.10), (2), $K \subset G$ and $\varphi \in \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ we deduce

$$\mathcal{U}_\varphi = \mathrm{span}_{\mathbb{C}}\{\varrho(k)\varphi \,|\, k \in K\} \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi.$$

From now on, let us suppose that (s1) $\rho : Q^- \to GL(\mathsf{V})$ is irreducible.

(4) is immediate from (1), (2), (s1) and Lemma 2.15.

(5) The arguments below will be similar to those in the proof of Lemma 2.16-(II). Let $\mathcal{V}_1$ be any closed $\varrho(G)$-invariant complex vector subspace of $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ with $\mathcal{V}_1 \neq \{0\}$. Here $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi \subset \mathcal{V}_{G/L}$ has the relative topology. By (1) and Weyl's theorem on semisimplicity of representations, there exists a closed $\varrho(G)$-invariant complex vector subspace $\mathcal{V}_2$ of $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$ such that

$$\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi = \mathcal{V}_1 \oplus \mathcal{V}_2.$$

Note here that both $\mathcal{V}_1$ and $\mathcal{V}_2$ are closed $\varrho(G)$-invariant complex vector subspaces of $\mathcal{V}_{G/L}$ by (2). If $\mathcal{V}_2 \neq \{0\}$, then Lemma 2.15 yields $\{0\} \neq \mathcal{W}_{G/L} \subset \mathcal{V}_1 \cap \mathcal{V}_2$, which is a contradiction. Hence one has $\mathcal{V}_2 = \{0\}$, and $\mathcal{V}_1 = \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi$. □

Proposition 3.5 leads to

Corollary 3.6. *Suppose that*

(s1) $\rho : Q^- \to GL(\mathsf{V})$ *is irreducible,*
(s4) $\mathfrak{g}$ *has a compact Cartan subalgebra, and*
(s5) $\triangle \subset \Lambda$.

Then, the following three items hold:

(I) $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$.
(II) ϱ *is an irreducible representation of* G *on* $\mathcal{V}_{G/L}$.
(III) *The representation module* $(\mathcal{V}_{G_{\mathbb{C}}/Q^-}, \tilde{\varrho})$ *is* G*-equivariant isomorphic to* $(\mathcal{V}_{G/L}, \varrho)$ *via* $\mathcal{F} : h \mapsto h|_{GQ^-}$.

Proof. By virtue of Lemma 2.16 and (5) on page 8, it suffices to prove that

ⓐ $$\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K < \infty.$$

That is evident when $(\mathcal{V}_{G/L})_K = \{0\}$. Hereafter, we investigate the case $(\mathcal{V}_{G/L})_K \neq \{0\}$. One can conclude this ⓐ, if we confirm the following item (D):

(D) For each $\phi \in (\mathcal{V}_{G/L})_K - \{0\}$, there exists a unique $\varrho(G)$-invariant complex vector subspace $\mathcal{V}(\phi)$ of $\mathcal{V}_{G/L}$ satisfying four conditions
  (1) $\dim_{\mathbb{C}} \mathcal{V}(\phi) < \infty$,
  (2) $\{0\} \neq \mathcal{W}_{G/L} \subset \mathcal{V}(\phi)$,
  (3) $\varrho : G \to GL\big(\mathcal{V}(\phi)\big)$ is irreducible, and
  (4) $\phi \in \mathcal{V}(\phi)$.

Indeed, fix a $\phi_0 \in (\mathcal{V}_{G/L})_K - \{0\}$ and take an arbitrary $\phi \in (\mathcal{V}_{G/L})_K - \{0\}$. Then, the vector spaces $\mathcal{V}(\phi_0)$ and $\mathcal{V}(\phi)$ both satisfy the conditions (1), (2), (3) and (4). It follows from (4), (2) and (3) that $\phi \in \mathcal{V}(\phi) = \mathcal{V}(\phi_0)$. Hence $(\mathcal{V}_{G/L})_K \subset \mathcal{V}(\phi_0)$, and it follows from (1) that $\dim_{\mathbb{C}}(\mathcal{V}_{G/L})_K \leq \dim_{\mathbb{C}} \mathcal{V}(\phi_0) < \infty$, so that ⓐ holds.

Now, let us confirm the (D) above. Take any $\phi \in (\mathcal{V}_{G/L})_K - \{0\}$. The uniqueness of $\mathcal{V}(\phi)$ comes from (2) and (3). From now on, we will prove its existence. The complex vector space $\mathcal{U}_\phi$ is $\varrho(K)$-invariant and $0 < \dim_{\mathbb{C}} \mathcal{U}_\phi < \infty$. Since the Lie group K is compact, there exist $\varrho(K)$-invariant complex vector subspaces $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{m_\phi}$ of $\mathcal{U}_\phi$ such that

① $$\mathcal{U}_\phi = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_{m_\phi},$$

and that $\mathcal{U}_j \neq \{0\}$ and $\varrho : K \to GL(\mathcal{U}_j)$ is irreducible $(1 \leq j \leq m_\phi)$. Then, for each $1 \leq j \leq m_\phi$, there exist a non-zero $\varphi_j \in \mathcal{U}_j$ and a linear function $\omega_j : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ such that $\varrho_*(H)\varphi_j = \omega_j(H)\varphi_j$ for all $H \in \mathfrak{t}_{\mathbb{C}}$. Here for every $1 \leq j \leq m_\phi$,

② $$\mathcal{U}_{\varphi_j} \overset{(2.10)}{=} \mathrm{span}_{\mathbb{C}}\{\varrho(k)\varphi_j \,|\, k \in K\} = \mathcal{U}_j$$

follows from $0 \neq \varphi_j \in \mathcal{U}_j$ and $\varrho : K \to GL(\mathcal{U}_j)$ being irreducible. Now, Proposition 3.5 and ② imply that

(i) $\dim_{\mathbb{C}} \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_j < \infty$,

(ii) $\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_j$ is a $\varrho(G)$-invariant complex vector subspace of $\mathcal{V}_{G/L}$,
(iii) $\mathcal{U}_j = \mathcal{U}_{\varphi_j} \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_j$, and
(iv) $\varrho : G \to GL\big(\varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_j\big)$ is irreducible

for each $1 \le j \le m_\phi$, and moreover,

$$\{0\} \neq \mathcal{W}_{G/L} \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_1 = \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_2 = \cdots = \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_{m_\phi}.$$

This, together with ① and (iii), yields

$$\phi \in \mathcal{U}_\phi = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_{m_\phi} \subset \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_1.$$

Setting $\mathcal{V}(\phi) := \varrho_*(U(\mathfrak{g}_{\mathbb{C}}))\varphi_1$, we conclude (D). □

3.3. **Case $\mathfrak{g}$ is a simple Lie algebra of Hermitian type.** Suppose $\mathfrak{g}$ to be a simple Lie algebra of Hermitian type. In this case, G/K is an irreducible Hermitian symmetric space of non-compact type, the action of G on G/K is effective, $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$, and there exists a non-zero elliptic element $W \in \mathfrak{g}$ such that the eigenvalue of $\operatorname{ad} W$ is $\pm i$ or zero and $K = C_G(W)$. Let us set

(3.7) $$K_{\mathbb{C}} := C_{G_{\mathbb{C}}}(W), \quad \mathfrak{p}^{\pm} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} W(A) = \pm iA\}, \quad P^{\pm} := \exp \mathfrak{p}^{\pm}$$

and remark that

(1) the element W is an H-element of $\mathfrak{g}$ relative to $\mathfrak{k}$, cf. Satake [14, p.54],
(2) the existence of W is unique up to sign $\pm$,
(3) G/K admits a unique G-invariant complex structure J' up to sign, which is induced by either $\operatorname{ad} W$ or $\operatorname{ad}(-W)$,
(4) Lemma 2.5 still holds even if one substitutes K, $K_{\mathbb{C}}$, P^s, $\mathfrak{p}^s$ and $K_{\mathbb{C}}P^s$ for L, $L_{\mathbb{C}}$, U^s, $\mathfrak{u}^s$ and Q^s, respectively ($s = \pm$),
(5) $\iota_{-s} : G/K \to G_{\mathbb{C}}/(K_{\mathbb{C}}P^{-s})$, $gK \mapsto g(K_{\mathbb{C}}P^{-s})$, is a G-equivariant holomorphic embedding (called the Borel embedding) where we fix the G-invariant complex structure on G/K induced by $\operatorname{ad}(sW)$ for each $s = \pm$,
(6) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$; $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^s] \subset \mathfrak{p}^s$, $[\mathfrak{p}^s, \mathfrak{p}^s] = \{0\}$ for each $s = \pm$,
(7) the representation $\operatorname{Ad}_{\mathfrak{p}^s} : K \to GL(\mathfrak{p}^s)$, $k \mapsto \operatorname{Ad} k|_{\mathfrak{p}^s}$, is irreducible ($s = \pm$),
(8) $W \in \mathfrak{t}$ follows by $\mathfrak{k} = \mathfrak{c}_{\mathfrak{g}}(W)$ and $\mathfrak{t}$ being a maximal torus of $\mathfrak{k}$,
(9) $\mathfrak{p}^+ = \bigoplus_{\beta \in \triangle(W,+)} \mathfrak{g}_\beta$, $\mathfrak{p}^- = \bigoplus_{\gamma \in \triangle(W,-)} \mathfrak{g}_\gamma$,
(10) $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\triangle(\mathfrak{k}) = \triangle(W, 0)$, $\triangle(\mathfrak{p}) = \triangle(W, +) \cup \triangle(W, -)$.

Here we refer to (2.2) and §§3.2 for $\triangle(W, \cdot\,)$ and $\triangle(\mathfrak{k})$, $\triangle(\mathfrak{p})$, respectively. Hereafter, we fix an H-element $W \in \mathfrak{g}$ relative to $\mathfrak{k}$ whenever $\mathfrak{g}$ is a simple Lie algebra of Hermitian type.

We first verify two Lemmas 3.8 and 3.10 below, and afterwards we add some setting in order to state Lemma 3.12, Proposition 3.13, etc.

Lemma 3.8. *Suppose that* (s4′) $\mathfrak{g}$ *is a simple Lie algebra of Hermitian type. For each $s = \pm$, we denote by $\mathfrak{m}^s$ the set of elements $A \in \mathfrak{p}^s$ which satisfy the following condition* (3.9):

(3.9) *for each $\phi \in (\mathcal{V}_{G/L})_K$ there exists an $n \in \mathbb{N}$ such that $\varrho_*(A)^n \phi = 0$.*

Then it turns out that for each $s = \pm$,

(i) $\mathfrak{m}^s$ *is an $\operatorname{Ad}_{\mathfrak{p}^s} K$-invariant complex vector subspace of $\mathfrak{p}^s$,*

(ii) $\mathfrak{m}^s = \mathfrak{p}^s$ *whenever* $\mathfrak{m}^s \neq \{0\}$. *In particular,* $\triangle(W, s) \subset \Lambda$ *if* $\mathfrak{m}^s \neq \{0\}$. *Here we refer to* §§3.1 *for* Λ.

Proof. (i) Take any $A, B \in \mathfrak{m}^s$ and $\alpha \in \mathbb{C}$. First, let us confirm that $\mathfrak{m}^s$ is a complex vector space. Since it is clear that $\alpha A \in \mathfrak{m}^s$, we only show that $A + B \in \mathfrak{m}^s$. Given a $\phi \in (\mathcal{V}_{G/L})_K$, there exist $n, k \in \mathbb{N}$ such that

$$\varrho_*(A)^n \phi = 0, \quad \varrho_*(B)^k \phi = 0$$

from $A, B \in \mathfrak{m}^s$. In addition, $\varrho_*(A)\varrho_*(B) = \varrho_*(B)\varrho_*(A)$ follows from $\mathfrak{m}^s \subset \mathfrak{p}^s$ and $[A, B] \in [\mathfrak{p}^s, \mathfrak{p}^s] = \{0\}$. These yield

$$\begin{aligned} \varrho_*(A+B)^{n+k}\phi &= \big(\varrho_*(A) + \varrho_*(B)\big)^{n+k}\phi \\ &= \sum_{j=0}^{n+k} \frac{(n+k)!}{j!(n+k-j)!} \varrho_*(A)^j \varrho_*(B)^{n+k-j}\phi \\ &= 0, \end{aligned}$$

and $A + B \in \mathfrak{m}^s$. Hence $\mathfrak{m}^s$ is a complex vector space. Now, let us prove that it is also $\mathrm{Ad}_{\mathfrak{p}^s} K$-invariant. Take an arbitrary $k \in K$. On the one hand, we have $\mathrm{Ad}\, k(A) \in \mathfrak{p}^s$. On the other hand, one has $\varrho(k^{-1})\phi \in (\mathcal{V}_{G/L})_K$, and there exists an $m \in \mathbb{N}$ such that $\varrho_*(A)^m(\varrho(k^{-1})\phi) = 0$ on GQ^-; then, for any $x \in GQ^-$ it follows from $k^{-1}x \in GQ^-$ that

$$0 = \big(\varrho_*(A)^m(\varrho(k^{-1})\phi)\big)(k^{-1}x) \overset{(2.8)}{=} \big(\varrho_*(\mathrm{Ad}\, k(A))^m \phi\big)(x).$$

That yields $\mathrm{Ad}\, k(A) \in \mathfrak{m}^s$, and thus $\mathfrak{m}^s$ is $\mathrm{Ad}_{\mathfrak{p}^s} K$-invariant.

(ii) $\mathfrak{m}^s = \mathfrak{p}^s$ comes from (i) and $\mathrm{Ad}_{\mathfrak{p}^s} : K \to GL(\mathfrak{p}^s)$, $k \mapsto \mathrm{Ad}\, k|_{\mathfrak{p}^s}$, being irreducible. $\triangle(W, s) \subset \Lambda$ comes from $\mathfrak{m}^s = \mathfrak{p}^s = \bigoplus_{\alpha \in \triangle(W,s)} \mathfrak{g}_\alpha$, (3.9) and (3.1). □

Lemma 3.10. *Suppose that* (s4′) $\mathfrak{g}$ *is a simple Lie algebra of Hermitian type. Then, there exists a fundamental root system* $\Pi_2 = \{\alpha_j\}_{j=1}^l$ *of* $\triangle = \triangle(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ *such that*

(1) $\alpha(-iW) \geq 0$ *for all* $\alpha \in \Pi_2$, *and*
(2) $\gamma(-iT) \geq 0$ *for every* $\gamma \in \Pi_2 \cap \triangle(W, 0)$.

Proof. It is enough to consider the lexicographic linear orderings on the dual space $(i\mathfrak{t})^*$ associated with ordered real bases $-iW =: C_1, C_2, \ldots, C_l$ and $-iW =: B_1, -iT =: B_2, B_3, \ldots, B_l$ of $i\mathfrak{t}$ when the sequence T, W of non-zero vectors in $\mathfrak{t}$ is linearly dependent and independent, respectively. □

3.3.1. *Additional setting.* Let us prepare for stating Lemma 3.12, Proposition 3.13, etc. With respect to the fundamental root system $\Pi_2 = \{\alpha_j\}_{j=1}^l$ in Lemma 3.10, we denote by $\tilde{\alpha} = \sum_{j=1}^l m_j \alpha_j$ ($\in \triangle$) the highest root and by $\{Z_j\}_{j=1}^l$ ($\subset i\mathfrak{t}$) the dual basis of $\{\alpha_j\}_{j=1}^l$. Then, there exists a unique index $1 \leq p \leq l$ such that

$$m_p = 1, \quad W = iZ_p$$

because $W \neq 0$, the eigenvalue of $\mathrm{ad}\, W$ is $\pm i$ or zero and $\alpha(-iW) \geq 0$ for all $\alpha \in \Pi_2$. Taking this p into account, we set

$$\Pi_0 := \Pi_2 - \{\alpha_p\} \tag{3.11}$$

and note that

(1) $\Pi_0 = \Pi_2 - \{\alpha_p\} = \Pi_2 \cap \triangle(W,0) \subset \triangle(\mathfrak{k})$,
(2) $\{\alpha_p\} = \Pi_2 \cap \triangle(W,+) \subset \triangle(\mathfrak{p})$, $\tilde{\alpha} \in \triangle(W,+) \subset \triangle(\mathfrak{p})$,
(3) $\triangle(W,+) = \{\beta = \sum_{j=1}^{l} n_j\alpha_j \in \triangle \,|\, n_p = 1\}$,
(4) both $\Pi_0 \cup \{\alpha_p\}$ and $\Pi_0 \cup \{-\tilde{\alpha}\}$ are fundamental root systems of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. cf. Appendix.

In the setting above, we will individually investigate two cases

$$\text{(C1) } \alpha_p(-iT) \leq 0 \leq \tilde{\alpha}(-iT), \quad \text{(C2) } 0 < \alpha_p(-iT) \text{ or } \tilde{\alpha}(-iT) < 0$$

and conclude $\dim_{\mathbb{C}} \mathcal{O}(G/L) = 1$, $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$ (resp. $\dim_{\mathbb{C}} \mathcal{O}(G/L) = \infty$) in the case (C1) (resp. (C2)).

3.3.2. (C1) $\alpha_p(-iT) \leq 0 \leq \tilde{\alpha}(-iT)$. We shall demonstrate Proposition 3.13 after

Lemma 3.12. *In the setting of* §§3.3.1*, suppose that* $\alpha_p(T) = 0$ *or* $\tilde{\alpha}(T) = 0$. *Then,* $\dim_{\mathbb{C}} \mathcal{V}_{G/L} = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$.

Proof. Let us recall that $\Pi_0 \subset \triangle(\mathfrak{k})$, and that both $\Pi_0 \cup \{\alpha_p\}$ and $\Pi_0 \cup \{-\tilde{\alpha}\}$ are fundamental root systems of $\triangle$, and remark that $\gamma(-iT) \geq 0$ for all $\gamma \in \Pi_0$.

Suppose $\alpha_p(T) = 0$. Then, $\alpha(-iT) \geq 0$ for all $\alpha \in \Pi_0 \cup \{\alpha_p\}$, and thus (2.4) holds for the fundamental root system $\Pi_0 \cup \{\alpha_p\}$. Moreover, $\beta \in \Pi_0 \cup \{\alpha_p\} - \triangle(T,0)$ implies $\beta \in \Pi_0 \subset \triangle(\mathfrak{k})$. Hence $\dim_{\mathbb{C}} \mathcal{V}_{G/L} = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$ by Theorem 2.17.

One can get the conclusion by considering the fundamental root system $\Pi_0 \cup \{-\tilde{\alpha}\}$ in case of $\tilde{\alpha}(T) = 0$. □

Proposition 3.13. *In the setting of* §§3.3.1*, suppose that*

(s1) $\rho : Q^- \to GL(\mathsf{V})$ *is irreducible, and*
(C1) $\alpha_p(-iT) \leq 0 \leq \tilde{\alpha}(-iT)$.

Then, the following three items hold:

(I) $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$.
(II) ϱ *is an irreducible representation of* G *on* $\mathcal{V}_{G/L}$.
(III) *The representation module* $(\mathcal{V}_{G_{\mathbb{C}}/Q^-}, \tilde{\varrho})$ *is* G*-equivariant isomorphic to* $(\mathcal{V}_{G/L}, \varrho)$ *via* $\mathcal{F} : h \mapsto h|_{GQ^-}$.

Proof. In case of $\alpha_p(T) = 0$ or $\tilde{\alpha}(T) = 0$, this proposition holds due to Lemmas 3.12 and 2.16. We aim to show that it holds even in case of $\alpha_p(-iT) < 0 < \tilde{\alpha}(-iT)$. From $\alpha_p, \tilde{\alpha} \in \triangle(W,+)$ and $\alpha_p(-iT) < 0 < \tilde{\alpha}(-iT)$ it follows that $\mathfrak{g}_{-\alpha_p} \in \mathfrak{p}^- \cap \mathfrak{u}^+$, $\mathfrak{g}_{\tilde{\alpha}} \in \mathfrak{p}^+ \cap \mathfrak{u}^+$, so that

$$E_{-\alpha_p} \in \mathfrak{m}^-, \quad E_{\tilde{\alpha}} \in \mathfrak{m}^+$$

by Proposition 3.2, (3.1) and (3.9). Consequently Lemma 3.8-(ii) yields $\triangle(\mathfrak{p}) = \triangle(W,-) \cup \triangle(W,+) \subset \Lambda$; furthermore, Lemma 3.4 yields $\triangle = \triangle(\mathfrak{k}) \cup \triangle(\mathfrak{p}) \subset \Lambda$. This and Corollary 3.6 enable us to achieve the aim. □

Proposition 3.13 leads to the two corollaries below.

Corollary 3.14. *In the setting of* §§3.3.1*, all holomorphic functions on* G/L *are constant if* (C1) $\alpha_p(-iT) \leq 0 \leq \tilde{\alpha}(-iT)$.

Proof. From Proposition 3.13-(III) and $\dim_{\mathbb{C}} \mathcal{O}(G_{\mathbb{C}}/Q^-) = 1$ one obtains

$$\dim_{\mathbb{C}} \mathcal{O}(G/L) = \dim_{\mathbb{C}} \mathcal{O}(G_{\mathbb{C}}/Q^-) = 1,$$

where we remark that $\mathcal{O}(G/L) = \mathcal{V}_{G/L}(\mathsf{V}, \rho)$, $\mathcal{O}(G_{\mathbb{C}}/Q^-) = \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$ and $\rho : Q^- \to GL(\mathsf{V})$ is irreducible in the case where $\mathsf{V} = \mathbb{C}$ and $\rho = \mathrm{id}$ (the trivial representation). $\square$

Corollary 3.15. *In the setting of* §§3.3.1*, suppose that*

(C1) $\alpha_p(-iT) \leq 0 \leq \tilde{\alpha}(-iT)$.

Then, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ *for every finite-dimensional complex vector space* V *and holomorphic homomorphism* $\rho : Q^- \to GL(\mathsf{V})$*.*

Proof. Take any finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$. If $\mathsf{V} = \{0\}$, then one has $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = 0 < \infty$. Henceforth, we investigate the case $\mathsf{V} \neq \{0\}$. By $0 < \dim_{\mathbb{C}} \mathsf{V} < \infty$ and Lemma 2.5.3 in [3, p.227] there exists a sequence of $\rho(Q^-)$-invariant complex vector subspaces

$$\{0\} \subsetneq \mathsf{V}_1 \subsetneq \cdots \subsetneq \mathsf{V}_{n-1} \subsetneq \mathsf{V}_n = \mathsf{V}$$

such that each quotient representation $\rho_{\mathsf{V}_i/\mathsf{V}_{i-1}} : Q^- \to GL(\mathsf{V}_i/\mathsf{V}_{i-1})$ is irreducible $(1 \leq i \leq n)$, where $\mathsf{V}_0 := \{0\}$. In this case, the supposition (C1) and Proposition 3.13-(I) allow us to assert that

① $$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_i/\mathsf{V}_{i-1}, \rho_{\mathsf{V}_i/\mathsf{V}_{i-1}}) < \infty \text{ for all } 1 \leq i \leq n.$$

Now, let $\mathrm{Pr}_i : \mathsf{V}_i \to \mathsf{V}_i/\mathsf{V}_{i-1}$ be the projection. Since $\mathrm{Pr}_i \circ \rho(q) = \rho_{\mathsf{V}_i/\mathsf{V}_{i-1}}(q) \circ \mathrm{Pr}_i$ for all $q \in Q^-$, the mapping $\mathcal{V}_{G/L}(\mathsf{V}_i, \rho) \ni \psi \mapsto \mathrm{Pr}_i \circ \psi \in \mathcal{V}_{G/L}(\mathsf{V}_i/\mathsf{V}_{i-1}, \rho_{\mathsf{V}_i/\mathsf{V}_{i-1}})$ is linear and its kernel accords with $\mathcal{V}_{G/L}(\mathsf{V}_{i-1}, \rho)$. Therefore it follows that

$$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_i, \rho) \big/ \mathcal{V}_{G/L}(\mathsf{V}_{i-1}, \rho) \leq \dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_i/\mathsf{V}_{i-1}, \rho_{\mathsf{V}_i/\mathsf{V}_{i-1}}),$$

and furthermore,

② $$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_i, \rho) \big/ \mathcal{V}_{G/L}(\mathsf{V}_{i-1}, \rho) < \infty \text{ for all } 1 \leq i \leq n$$

by ①. Here $\mathcal{V}_{G/L}(\mathsf{V}_i, \rho) \big/ \mathcal{V}_{G/L}(\mathsf{V}_{i-1}, \rho)$ is the quotient vector space of $\mathcal{V}_{G/L}(\mathsf{V}_i, \rho)$ by $\mathcal{V}_{G/L}(\mathsf{V}_{i-1}, \rho)$. By the mathematical induction on i together with ② and $\mathsf{V}_0 = \{0\}$, one can confirm that $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_i, \rho) < \infty$ for all $1 \leq i \leq n$. Thus we have $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}_n, \rho) < \infty$. $\square$

3.3.3. (C2) $0 < \alpha_p(-iT)$ or $\tilde{\alpha}(-iT) < 0$. We first prove the following lemma from which we deduce Proposition 3.17:

Lemma 3.16. *In the setting of* §§3.3.1*, the following four items hold*:

(i1) $\mathfrak{p}^+ \subset \mathfrak{u}^+$ *if* $0 < \alpha_p(-iT)$.
(i2) $\mathfrak{p}^- \subset \mathfrak{u}^+$ *if* $\tilde{\alpha}(-iT) < 0$.
(ii1) $L \subset K$, $Q^- \subset K_{\mathbb{C}} P^-$ *if* $0 < \alpha_p(-iT)$.
(ii2) $L \subset K$, $Q^- \subset K_{\mathbb{C}} P^+$ *if* $\tilde{\alpha}(-iT) < 0$.

Proof. For any $\beta \in \triangle(W,+)$ we have

① $$\alpha_p(-iT) \leq \beta(-iT) \leq \tilde{\alpha}(-iT)$$

because $\triangle(W,+) = \{\sum_{j=1}^{l} n_j\alpha_j \in \triangle \mid n_p = 1\}$ and $\gamma(-iT) \geq 0$ for all $\gamma \in \Pi_0$.

(i1) Suppose that $0 < \alpha_p(-iT)$. Then, it follows from ① and $\mathfrak{p}^+ = \bigoplus_{\beta\in\triangle(W,+)} \mathfrak{g}_\beta$ that $\mathfrak{p}^+ \subset \mathfrak{u}^+$.

(i2) Suppose $\tilde{\alpha}(-iT) < 0$. For any $\beta' \in \triangle(W,-)$, there exists a $\beta \in \triangle(W,+)$ such that $\beta' = -\beta$. Then, the supposition and ① yield $0 < -\tilde{\alpha}(-iT) \leq \beta'(-iT)$. Therefore we conclude $\mathfrak{p}^- \subset \mathfrak{u}^+$ by $\mathfrak{p}^- = \bigoplus_{\beta'\in\triangle(W,-)} \mathfrak{g}_{\beta'}$.

(ii1) Suppose that $0 < \alpha_p(-iT)$. Then one obtains

$$\mathfrak{p}^+ \subset \mathfrak{u}^+$$

from (i1). We know that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- = \mathfrak{u}^+ \oplus \mathfrak{q}^-,$$

and that $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$ (resp. $\mathfrak{q}^-$) coincides with the orthogonal complement of $\mathfrak{p}^+$ (resp. $\mathfrak{u}^+$) with respect to $B_{\mathfrak{g}_{\mathbb{C}}}$. Hence $\mathfrak{p}^+ \subset \mathfrak{u}^+$ gives rise to

$$\mathfrak{q}^- \subset \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-.$$

This yields $Q^- \subset K_{\mathbb{C}}P^-$, because Q^- and $K_{\mathbb{C}}P^-$ are connected closed subgroups of $G_{\mathbb{C}}$ whose Lie algebras are $\mathfrak{q}^-$ and $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$, respectively. Moreover, $Q^- \subset K_{\mathbb{C}}P^-$ yields $L = G \cap Q^- \subset G \cap (K_{\mathbb{C}}P^-) = K$.

(ii2) One can conclude (ii2) by proof similar to that of (ii1). □

Now, we are in a position to deduce

Proposition 3.17. *In the setting of* §§3.3.1*, the following conditions* (a)*,* (b)*,* (c) *and* (C2) *are equivalent*:

(a) *L is included in the maximal compact subgroup K of G, and there exists a G-invariant complex structure J' on G/K so that the fibering of G/L by K/L over G/K is holomorphic.*

(b) $\dim_{\mathbb{C}} \mathcal{O}(G/L) = \infty$.

(c) $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$*, i.e., G/L has a non-constant holomorphic function.*

(C2) $0 < \alpha_p(-iT)$ *or* $\tilde{\alpha}(-iT) < 0$.

Proof. (a)⇒(b). Suppose that (a) holds, and let $\mathrm{Pr} : G/L \to G/K$, $gL \mapsto gK$. Then Pr is holomorphic. Since the Hermitian symmetric space G/K is biholomorphic to a domain in some $\mathbb{C}^n$, it is natural that

$$\dim_{\mathbb{C}} \mathcal{O}(G/K) = \infty.$$

The mapping $\mathcal{O}(G/K) \ni f \mapsto f \circ \mathrm{Pr} \in \mathcal{O}(G/L)$ is injective linear, since $\mathrm{Pr} : G/L \to G/K$ is surjective holomorphic. Accordingly $\infty = \dim_{\mathbb{C}} \mathcal{O}(G/K) \leq \dim_{\mathbb{C}} \mathcal{O}(G/L)$.

(b)⇒(c) is clear.

(c)⇒(C2). This is the contraposition of Corollary 3.14.

(C2)⇒(a). First, let us suppose $0 < \alpha_p(-iT)$, and fix the G-invariant complex structure J' on G/K induced by $\mathrm{ad}\, W$. Then, $\iota_- : G/K \to G_{\mathbb{C}}/(K_{\mathbb{C}}P^-)$, $gK \mapsto g(K_{\mathbb{C}}P^-)$, is a biholomorphism of G/K onto a domain in $G_{\mathbb{C}}/(K_{\mathbb{C}}P^-)$. Moreover, Lemma 3.16-(ii1) assures that $L \subset K$, $Q^- \subset K_{\mathbb{C}}P^-$, and thus the mappings $\mathrm{Pr} : G/L \to G/K$, $gL \mapsto gK$, and $\mathrm{Pr}_- : G_{\mathbb{C}}/Q^- \to G_{\mathbb{C}}/(K_{\mathbb{C}}P^-)$, $aQ^- \mapsto a(K_{\mathbb{C}}P^-)$, are both well-defined; besides, Pr_- is holomorphic and $\mathrm{Pr}_-\big(\iota(G/L)\big) \subset \iota_-(G/K)$.

$$\begin{array}{ccc} G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^- \\ \big\downarrow \mathrm{Pr} & & \big\downarrow \mathrm{Pr}_- \\ G/K & \xrightarrow{\ \iota_-\ } & G_{\mathbb{C}}/(K_{\mathbb{C}}P^-) \end{array}$$

Thus $\mathrm{Pr} : G/L \to G/K$ is holomorphic because of $\mathrm{Pr} = (\iota_-)^{-1} \circ \mathrm{Pr}_- \circ \iota$.

In case the $\tilde{\alpha}(-iT) < 0$ also, one can conclude that $L \subset K$ and $\mathrm{Pr} : G/L \to G/K$ is holomorphic by Lemma 3.16-(ii2) with fixing the complex structure J' on G/K induced by $\mathrm{ad}(-W)$. □

3.3.4. *Summary.* Let us prove Theorem 1.2.

Proof of Theorem 1.2. Suppose $\mathfrak{g}$ to be a simple Lie algebra of Hermitian type. Then, there are two cases where

(C1) $\alpha_p(-iT) \le 0 \le \tilde{\alpha}(-iT)$;
(C2) $0 < \alpha_p(-iT)$ or $\tilde{\alpha}(-iT) < 0$.

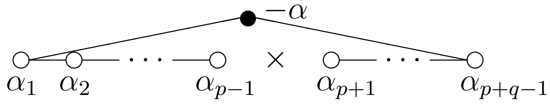
Here we refer to §§3.3.1 for $\alpha_p, \tilde{\alpha}$. Needless to say, the cases (C1) and (C2) are mutually exclusive and one of them necessarily occurs. In these cases, two Corollaries 3.14 and 3.15, and Proposition 3.17 imply that

(C1) $\dim_{\mathbb{C}} \mathcal{O}(G/L) = 1$, and $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ for every finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$;
(C2) $\dim_{\mathbb{C}} \mathcal{O}(G/L) = \infty$.

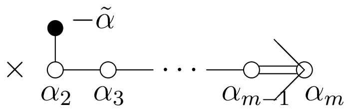
Hence, we can get the conclusion. □

Appendix. The following diagrams show that $\Pi_0 \cup \{-\tilde{\alpha}\}$ are fundamental root systems of $\triangle$. cf. §§3.3.1. Here we take the Dynkin diagrams from Bourbaki [4].

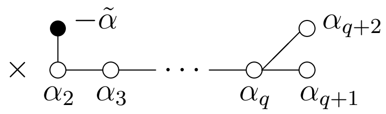
(AIII) $\mathfrak{su}(p,q)$ with $p, q \ge 1$, $W = iZ_p$, $\tilde{\alpha} = \sum_{j=1}^{p+q-1} \alpha_j$:



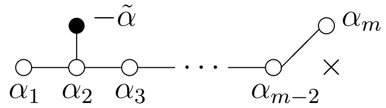
(BI) $\mathfrak{so}(2, 2m-1)$ with $m \ge 3$, $W = iZ_1$, $\tilde{\alpha} = \alpha_1 + 2\sum_{k=2}^{m} \alpha_k$:



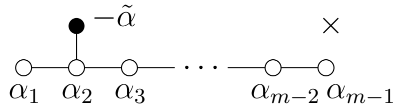
(DI) $\mathfrak{so}(2, 2q+2)$ with $q \ge 1$, $W = iZ_1$, $\tilde{\alpha} = \alpha_1 + 2\sum_{k=2}^{q} \alpha_k + \alpha_{q+1} + \alpha_{q+2}$:



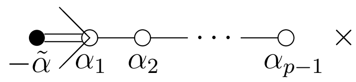
(DIII) $\mathfrak{so}^*(2m)$ with $m \ge 3$, $W = iZ_{m-1}$, $\tilde{\alpha} = \alpha_1 + 2\sum_{k=2}^{m-2} \alpha_k + \alpha_{m-1} + \alpha_m$:



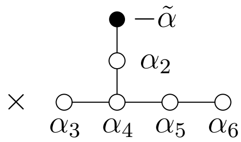
$W = iZ_m$:



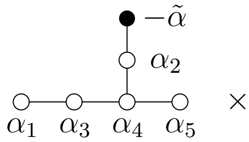
(CI) $\mathfrak{sp}(p, \mathbb{R})$ with $p \geq 1$, $W = iZ_p$, $\tilde{\alpha} = 2\sum_{j=1}^{p-1} \alpha_j + \alpha_p$:



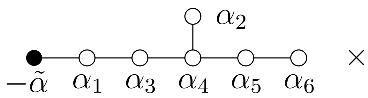
(EIII) $\mathfrak{e}_{6(-14)}$, $W = iZ_1$, $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$:



$W = iZ_6$:



(EVII) $\mathfrak{e}_{7(-25)}$, $W = iZ_7$, $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$:



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